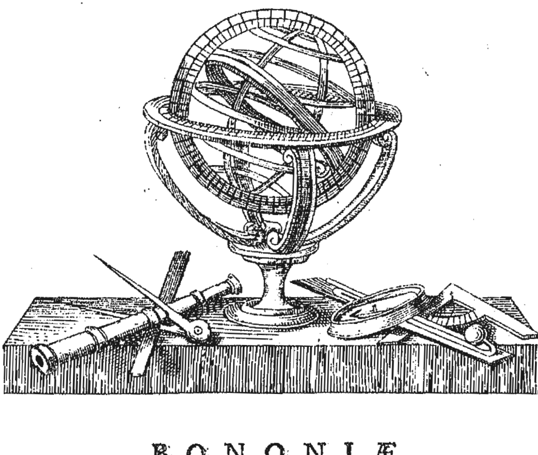
VINCENTII RICCATI
SOC. JESU
OPUSCULORUM
Ad res Physicas, & Mathematicas pertinentium
TOMUS SECUNDUS.



BONONIÆ

EX TYPOGRAPHIA SANCTI THOMÆ AQUINATIS

MDCCLXII.

SUPERIORUM AUCTORITATE.

PRÆFATIO.

QUem diutius fortaſſe, quam par eſt, expectaſti, Benevole Lector, meorum Opuſculorum Tomum alterum, ne pergeres fruſtra expectare, aliis typis impreſſum in præſentia tibi exhibeo. Satius viſum eſt, aliquam in tomis diſſimilitudinem pati, quam abuti diutius patientiâ tuâ. Si eâ humanitate, qua primum excepiſti, excipias alterum, ad tertium quamprimùm edendum quodammodo provocabis.

Opuſcula

Opuſcula duo jamdiu in publicam lucem prodierunt, nimirum diſquiſitio analytica de integratione formulæ $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$ per arcus ellypticos, & hyperbolicos, quæ italico idomate impreſſa eſt in Collectione Lucenſi; tum animadverſiones in formulam differentialem, in qua indeterminatæ ad unicam tantum dimenſionem aſcendunt; quæ diſquiſitio edita eſt in tertia parte ſecundi tomi Academiæ Bononienſis. Verùm huic aliquot additamenta appoſui, quæ, ut ipſe cognoſces, maximæ erunt utilitati. Reliqua omnia inedita ſunt antea, & nunc primum in lucem proferuntur.

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ANIMADVERSIONES

In formulam differentialem, in qua indeterminatæ ad unicam tantum dimenſionem aſcendunt.

Diſquiſitio Mathematica. (*a*)

I. Exiſtunt plerique homines ſane docti, & in rebus analyticis multum verſati, qui arbitrantur, in formula differentiali, in qua indeterminatæ ad unicam ſolumodo dimenſionem aſcendunt, quæque ita œcumenice exprimi poteſt $\overline{ax+b+cy}.dy=\overline{fx+g+hy}.dx$, ſemper indeterminatas poſſe a ſe invicem ſeparari, ut illis ſeparatis ad geometricam conſtructionem deveniatur. Verum ſi illis dumtaxat utamur methodis, quæ hactenus traditæ ſunt ab Analyſtis, puto caſum adeſſe, in quo indeterminatarum ſeparatio nequaquam obtinetur. Ex hiſce methodis illas, quæ elengantiores nobis videntur, breviter exponemus, & primum eam formulam œcumenicam eliciemus, in qua non ſolum incognitæ ſeparatæ inveniuntur, ſed etiam integralis differentialis formulæ exhibetur, & res omnis ad exponentialium calculum traducitur.

II. Hujuſmodi formula integralis longo quidem, ſed non difficili calculo inveniri poteſt per methodum integrandi ſine prævia ſeparatione, qua uſus eſt Joannes Bernoullius vir toto orbe terrarum clarus in Ac. Petr. T. pr. Expoſitæ formulæ integralis ſupponatur eſſe

$\overline{Ax+B+Cy}^{p+q}=M.\overline{Fx+G+Hy}^{p-q}$, in qua

A   A, B,

(*a*) *Diſquiſitio hæc ſimul cum additamento primo typis emiſſa eſt in tertia parte ſecundi tomi Academiæ Bononienſis anno* 1747.

A, B, C, F, G, H, p, q sunt quantitates constantes, & indeterminatæ deinceps per analytica subsidia determinandæ. Quantitas M est constans quæpiam. A numeris transitus fiat ad logarithmos, & orietur formula.

$$\overline{p+q}.l\,\overline{Ax+B+Cy} = l\,M + \overline{p-q}.l\,\overline{Fx+G+Hy}$$

In hac differentiæ accipiantur, ut fiat

$$\frac{\overline{p+q}.A\,dx + \overline{p+q}.C\,dy}{Ax+B+Cy} = \frac{\overline{p-q}.F\,dx + \overline{p-q}.H\,dy}{Fx+G+Hy}$$

Æquatio a divisoribus liberata hanc formam accipit

$$\begin{array}{lcl} \overline{p+q}.AFx\,dx + \overline{p+q}.CFx\,dy & & \overline{p-q}.AFx\,dx + \overline{p-q}.AHx\,dy \\ \overline{p+q}.AG\;dx + \overline{p+q}.CG\;dy & = & \overline{p-q}.BF\;dx + \overline{p-q}.BH\;dy \\ \overline{p+q}.AHy\,dx + \overline{p+q}.CHy\,dy & & \overline{p-q}.CFy\,dx + \overline{p-q}.CHy\,dy; \end{array}$$

quæ, ut collatio commode institui possit, in hunc modum disponenda est

$$\begin{array}{l} \overline{p+q}.CFx + \overline{p+q}.CG + \overline{p+q}.CHy \\ -\overline{p+q}.AHx - \overline{p+q}.BH - \overline{p+q}.CHy \end{array} .dy = \begin{array}{l} \overline{p-q}.AFx + \overline{p-q}.BF + \overline{p-q}.CFy \\ -\overline{p-q}.AFx - \overline{p-q}.AG - \overline{p-q}.AHy \end{array} .dx$$

III. Jam vero hæc ultima æquatio comparetur cum æquatione data, quæ legitur num. I.; orientur æquationes sex, ex quibus constantium indeterminatarum valores licebit determinare.

1) $\overline{p+q}.CF - \overline{p+q}.AH = a$	4) $-2q.AF = f$
2) $\overline{p+q}.CG - \overline{p+q}.BH = b$	5) $\overline{p-q}.BF - \overline{p-q}.AG = g$
3) $2q.CH = c$	6) $\overline{p-q}.CF - \overline{p-q}.AH = h$.

Quando sex sunt coefficientes determinandi, & duo exponentes, constat, cum de coefficientibus, tum de exponentibus unum posse pro arbitratu determinari.

IV. Ut exponentes p, & q definiantur, hujusmodi ineatur calculus. Multiplica æquationem tertiam per quartam, & habebis.

7) $CAFH = \frac{-fc}{4qq}$. Adde primam, & sextam

8) $2p.CF - 2p.AH = a+h$, sive $CF - AH = \frac{a+h}{2p}$.

Detrahe sextam a prima, & erit

9) $2q.CF + 2q.AH = a-h$, sive $CF + AH = \frac{a-h}{2q}$.

Addita

Addita octava, & nona proveniet

10) $2\,CF = \frac{a+b}{2p} + \frac{a-b}{2q}$. Deducta autem octava ex nona orietur

11) $2\,AH = \frac{a-b}{2q} - |\frac{a+b}{2p}$. Multiplicetur jam decima per undecimam, & fiet

$$4\,ACFH = \frac{a^2-b^2}{4pq} + \frac{\overline{a-b}^2}{4qq} - |\frac{aa-bb}{4pq} - |\frac{\overline{a+b}^2}{4pp}$$

sivè $4\,ACFH = \frac{\overline{a-b}^2}{4qq} - |\frac{\overline{a+b}^2}{4pp}$: sed ex septima

$4\,ACFH = \frac{-fc}{qq}$: Ergo

$$\frac{-4fc}{qq} = \frac{\overline{a-b}^2}{qq} - |\frac{\overline{a+b}^2}{pp} \text{ sivè } \frac{\overline{a+b}^2}{pp} = \frac{\overline{a-b}^2}{q^2} + \frac{4fc}{qq}$$

Igitur $p : q :: a+b : \sqrt{\overline{a-b}^2 + 4fc}$. Quare si ponatur $p = a+b$ erit $q = \sqrt{\overline{a-b}^2 + 4fc}$: quæ duæ æquationes valores exponentium determinant.

V. Antequam progredior, opportunum judico definire valores G, B ex aliis constantibus indeterminatis. Ex æquatione secunda erit

$B = \frac{\overline{p+q}.CG - b}{\overline{p-q}.H}$: & ex quinta

$B = \frac{\overline{p+q}.AG + g}{\overline{p-q}.F}$. Igitur

$\frac{\overline{p+q}.CG - b}{H} = \frac{\overline{p+q}.AG + g}{F}$. Quare

$\overline{p+q}.\overline{CFG - AHG} = bF + gH$: Igitur

$G = \frac{bF + gH}{\overline{p+q}.\overline{CF - AH}}$: atqui ex decima, & undecima constat $CF - AH = \frac{a+b}{2p} = \frac{1}{2}$ substituto valore p: Ergo

$$G = \frac{2bF + 2gH}{p+q} = \frac{2bF + 2gH}{a + b + \sqrt{\overline{a-b}^2 + 4fc}}$$ si pro p, & q valores inventos N. IV. substituas.

Simili ratione, & calculo invenies valorem B, scilicet

$$B = \frac{-2bA - 2gC}{a + b - \sqrt{\overline{a-b}^2 + 4fc}}.$$

VI. Hisce inventis ex quarta nancisceris

$$F = \frac{-f}{2Aq} = \frac{-f}{2A\sqrt{\overline{a-b}^2 + 4fc}}.$$ Item ex decima

$$F = \frac{a+b}{4Cp} + \frac{a-b}{4Cq} = \frac{a - b + \sqrt{\overline{a-b}^2 + 4fc}}{4C\sqrt{\overline{a-b}^2 + 4fc}}.$$ Igitur

$$-\frac{f}{A} = \frac{a - b + \sqrt{\overline{a-b}^2 + 4fc}}{2C}$$: Ergo

$$A : C :: \frac{-f}{a - b + \sqrt{\overline{a-b}^2 + 4fc}} : \tfrac{1}{2}.$$ Igitur si fiat

$$A = \frac{-f}{a - b + \sqrt{\overline{a-b}^2 + 4fc}}$$ erit

$C = \frac{1}{2}$. Tum ex quarta

$$F = \frac{a - b + \sqrt{\overline{a-b}^2 + 4fc}}{2\sqrt{\overline{a-b}^2 + 4fc}}$$; et ex secunda

$$H = \frac{c}{2\sqrt{\overline{a-b}^2 + 4fc}}.$$ Hosce valores introduc in formulas exprimentes valores B, G, quæ habentur N. V., & erit

$$G = \frac{b \cdot \overline{a - b + \sqrt{\overline{a-b}^2 + 4fc}} + 2gc}{\overline{a + b + \sqrt{\overline{a-b}^2 + 4fc}} \cdot \sqrt{\overline{a-b}^2 + 4fc}},$$ &

B =

$$B = \frac{\overline{2bf-g}\cdot\overline{a-b+\sqrt{\overline{a-b}^2+4fc}}}{\overline{a+b-\sqrt{\overline{a-b}^2+4fc}}\cdot\overline{a-b+\sqrt{\overline{a-b}^2+4fc}}}.$$ Itaque hisce valoribus substitutis habebitur integralis formulæ datæ.

$$\left|\frac{-fx}{a-b+\sqrt{\overline{a-b}^2+4fc}}+\frac{\overline{2bf-g}\cdot\overline{a-b+\sqrt{\overline{a-b}^2+4fc}}}{\overline{a+b-\sqrt{\overline{a-b}^2+4fc}}\cdot\overline{a-b+\sqrt{\overline{a-b}^2+4fc}}}+\tfrac{1}{2}y\right|^{a+b+\sqrt{\overline{a-b}^2+4fc}}.$$

$$M.\left|\frac{x\cdot\overline{a-b+\sqrt{\overline{a-b}^2+4fc}}}{2\sqrt{\overline{a-b}^2+4fc}}+\frac{b\cdot\overline{a-b+\sqrt{\overline{a-b}^2+4fc}}+2gc}{\overline{a+b+\sqrt{\overline{a-b}^2+4fc}}\cdot\sqrt{\overline{a-b}^2+4fc}}+\frac{cy}{\sqrt{\overline{a-b}^2+4fc}}\right|^{a+b-\sqrt{\overline{a-b}^2+4fc}}$$

VII. Hujusmodi integralem formulam non attente consideranti videri facile poterit, proposita differentialis formula integrationem semper admittere, aut ad formulam reduci prædictam exponentibus constantibus. Verum si recte advertes, invenies plures esse casus, ad quos integralis formula nequaquam pertinet.

VIII. Primo omnium pone $\overline{a-b}^2 = -4fc$, ita ut $\sqrt{\overline{a-b}^2+4fc} = o$, quæ suppositio applicata inventæ integragrali, dabit $\frac{x \cdot \overline{a-b}}{2} + \frac{b \cdot \overline{a-b} + 2gc}{a+b} + cy = 0$. Hanc æquationem satisfacere propositæ æquationi differentiali, experienti palam fiet. Verum, ut paullo infra, nempe N. XV. constabit, aliam methodum in usum traducens plane cognovi, præter lineam rectam, quæ ab ultima æquatione exprimitur, alias quoque curvas ad nostram differentialem pro hac hypothesi pertinere.

IX. Deinceps si $\sqrt{\overline{a-b}^2 + 4fc}$ sit quantitas imaginaria, quod eveniet, ubi alterutra ex speciebus f, c sit negativa, & rectangulum fc sit majus quarta parte quadrati $\overline{a-b}$, formula omnis cum in exponentibus, tum in coefficientibus imaginariis abundat, quas quomodo expellas, non video, neque fortasse expellere poteris, nisi ad formulam differentialem regrediaris. Quapropter pro hac hypothesi integralis inventa est prorsus inutilis, neque ad ullam nos constructionem perducit.

X. Hæc autem ab aliis animadversa fuisse non ignoramus; & vulgo notum est, in suppositione N. VIII. integrationem formulæ ad Hyperbolæ quadraturam pertinere, in suppositione secunda N. IX. etiam ad circuli quadraturam: quam rem alia adhibita methodo patefaciam. Verum alia adest suppositio, in qua res nondum perfecta est, neque constat quo pacto ex formula differentiali curvam construamus. Ea autem est, quum $fc = ab$; qua in hypothesi constat $\sqrt{\overline{a-b}^2 + 4fc} = a + b$. Quid autem in hac hypothesi eveniet nostræ integrali? Nimirum illa rite operatione instituta in hanc mutabitur

$$\frac{2bf - 2ga}{0 \cdot a} + y = \frac{M^{\frac{-1}{2a+2b}} + fx}{a}$$, quæ ad nullam deducere constructionem potest, propterea quod in illa addenda est y quantitati constanti infinitæ. Quæ hypothesis casum complectitur absolute integrabilem, quum scilicet $a = -b$, seu $a + b = 0$.

XI. Quæ cum ita sint tametsi integralis inventa sæpenumero utilitati esse possit, tamen in illis casibus, quos illa nequa-

quam

quam attingit, alia methodo uti, necessarium est. Quapropter singulatim omnia contemplans ajo primo, æquationem ubique fore integrabilem, quotiescumque $a = -b$. Nam transpositis terminis erit $a.\overline{xdy + ydx} + bdy + cydy = fxdx + gdx$, quæ integrata, dat $A + axy + by + \frac{cy^2}{2} = \frac{fx^2}{2} + gx$.

Hanc autem æquationem ad conicas sectiones pertinere, nemo unus est, qui ignoret; imo si $fc = ab$, de quo casu mentionem fecimus N. X., sive in hac hypothesi $fc = -aa$, erit ad parabolam. Quantitas A est constans addita in integratione.

XII. Deinceps ajo, si $fb = ag$ obtineri indeterminatarum separationem ope substitutionis $fx + g + by = zy$. Namque transpositis terminis erit $x = \frac{-g - by + zy}{f}$.

Igitur differentiando $dx = \frac{-bdy + zdy + ydz}{f}$, factisque opportunis substitutionibus nostra formula mutabitur in hanc

$$-\frac{ga}{f}dy - \frac{ab}{f}ydy + \frac{azydy}{f} = -\frac{bzdy}{f} + \frac{z^2ydy}{f} + \frac{y^2zdz}{f}$$
$$+ bdy + cydy$$

Deletis autem duobus primis terminis, qui ex suppositione $fb = ag$ destruuntur, factisque necessariis operationibus formula in hanc mutabitur

$$\overline{ab - cf}.-ydy - \overline{a - b}.-zydy + z^2.-ydy = y^2zdz:$$

sive $\frac{-dy}{y} = \frac{zdz}{zz - z.\overline{a + b} + ab - cf}$, in qua inveniuntur incognitæ separatæ. Si sit $a = -b$ utrumque æquationis membrum ad logarithmos pertinebit, ex quibus si ad numeros fiat transitus, invenietur æquatio numeri superioris nostræ hypothesi accommodata.

XIII. Ut inventa formula ad magis notas redigatur, fiat $zz - z.\overline{a + b} + \frac{\overline{a + b}^2}{4} = tt$. Igitur $z - \frac{a - b}{2} = t$, & $dz = dt$, factisque necessariis substitutionibus obtinebimus

$-dy$

$$-\frac{dy}{y} = \frac{tdt + dt.\overline{\frac{a+b}{2}}}{tt - \left|\frac{\overline{a+b}^2}{4} + ab - cf\right.} = \frac{tdt + dt.\overline{\frac{a+b}{2}}}{tt - \left|\frac{\overline{a-b}^2}{4} - cf\right.}.$$

XIV. Tres jam casus oportet distinguere. Aut enim $\frac{\overline{a-b}^2}{4} + cf$ est quantitas positiva, aut $= o$, aut negativa. Sit primo quantitas affirmativa, eaque fiat $= nn$. Item fiat $\frac{a+b}{2} = m$. Quare formula in hanc mutabitur.

$$-\frac{dy}{y} = \frac{tdt + mdt}{tt - nn} = \frac{m+n}{2n}.\frac{dt}{t-n} + \frac{n-m}{2n}.\frac{dt}{t+n}.$$

Quæ omnia integrata per logarithmos dant

$lA - ly = \frac{m+n}{2n}\, l\,\overline{t-n} + \frac{n-m}{2n}\, l\,\overline{t+n}$, & facto transitu a logarithmis ad numeros $\frac{A}{y} = \overline{t-n}^{\frac{m+n}{2n}} . \overline{t+n}^{\frac{n-m}{2n}}$.

Huic autem formulæ si recte substitutiones adhibeantur, proveniet

$$A^{2n} = \overline{fx + g + \frac{b-a}{2}y - ny}^{\,m+n} . \overline{fx + g + \frac{b-a}{2}y + ny}^{\,n-m}$$

non dissimilis illi, quam per primam methodum universalius invenimus N. VI. In hac itaque hypothesi curva aut algebraica est, aut exponentialis.

XV. Ad alterum casum accedo, in quo $\frac{\overline{a-b}^2}{4} + cf = o$. In hoc formula ita sese habet

$$-\frac{dy}{y} = \frac{tdt + mdt}{t^2} \text{ sive } \frac{dy}{y} + \frac{dt}{t} = \frac{-mdt}{t^2}.$$

Huic formulæ, ut ab aliis demonstratum est, duplex æquatio satisfacit, altera, quæ sine integratione, altera, quæ per inte-

gratio-

grationem obtinetur. Prima est $y\,t = o$, sive per substitutiones regrediendo $y\,z - y \cdot \frac{a+b}{2} = o$, sive

$$fx + g + by - y \cdot \frac{a+b}{2} = fx + g + y \cdot \frac{\overline{b-a}}{2} = o$$

quæ prorsus eadem est cum illa, quam per primam methodum invenimus N. VIII, dummodo præsenti accommodetur hypothesi. Altera, quæ per integrationem obtinetur, est hujusmodi $-lA + ly + lt = \frac{m}{t}$, quæ dependet a sola hyperbolæ quadratura.

XVI. Demum si $\frac{\overline{a-b}^2}{4} + cf$ est quantitas negativa, hoc est $= -nn$, hoc pacto disponatur æquatio

$$\frac{-dy}{y} = \frac{t\,dt}{tt+nn} + \frac{m}{n} \cdot \frac{n\,dt}{tt+nn}$$: quæ integrata exhibet

$lA - ly = l\sqrt{tt+nn} + \frac{m}{n}$ in arcum circularem, cujus radius $= n$, tangens $= t$. Quapropter in hoc casu æquatio dependet cum a circuli, tum ab hyperbolæ quadratura.

XVII. Præterea ajo, in suppositione $cg = bb$ obtineri indeterminatarum separationem, si utaris substitutione $ax + b + cy = zx$, & indeterminatam x ab æquatione expellas. Quoniam idem est calculus, eademque consectaria ac in hypothesi superiori, libenter omnia omitto, atque analystis relinquo.

XVIII. Postea ajo, ubi nulla ex prædictis æqualitatibus locum habeat, faciendam esse prius hujusmodi substitutionem $y = z + \frac{fb - ag}{ab - fc}$, atque adeo $dy = dz$. Hac autem facta proveniet æquatio

$$\begin{matrix} axdz + bdz \\ + \frac{c \cdot \overline{fb-ag}}{ab-fc} \cdot dz \end{matrix} + zdz = \begin{matrix} fxdx + gdx \\ + \frac{b \cdot \overline{fb-ag}}{ab-fc}\,dx \end{matrix} + bzdx,$$

in qua adsunt conditiones secundæ hypothesis, quæ N. XII habetur, ut patebit consideranti æquales esse hujusmodi quantitates $-\frac{fb}{}$

$\frac{fb + fc \cdot \overline{fb - ag}}{ab - fc}$, $\frac{ag \pm ab \cdot \overline{fb - ag}}{ab - fc}$. Proinde per methodum ſecundæ hypotheſis conſtruetur, & eadem prorſus elicientur conſectaria.

XIX. Denique ajo, per ſubſtitutionem $x = z + \frac{cg - bb}{ab - fc}$ oriri æquationem præditam conditione hypotheſis tertiæ, nempe N. XVII. atque adeo in ſuis indeterminatis ſeparabilem.

XX. Obtinuimus itaque indeterminatarum ſeparationem in omnibus caſibus illo dumtaxat excepto, ubi $ab = fc$: quo in caſu niſi adſit altera ex conditionibus, quæ continentur in ſecunda, & tertia hypotheſi, ſubſtitutio, qua utimur, implicat quantitates infinitas, atque adeo analyſeos ſubſidia inutilia reddit.

XXI. Ad ſeparandas indeterminatas in propoſita formula poſſem quoque alia methodo uti, quam paucis exponam. Fiat $x = z + A$, $y = u + B$. A, B ſunt quantitates conſtantes determinandæ in operationis progreſſu. Factis ſubſtitutionibus oritur æquatio

$$\begin{aligned} azdu + aAdu + cudu &= fzdz + fAdz + budz. \\ + bdu \quad\quad\quad &\quad\quad + gdz \\ + cBdu \quad\quad\quad &\quad\quad + bBdz \end{aligned}$$

Per determinationem duarum conſtantium A, B, ejicio ab æquatione utriuſque partis ſecundos terminos, ponens $aA + b + cB = o$, & $fA + g + bB = o$: ex quibus oritur $A = \frac{cg - bb}{ab - fc}$, $B = \frac{fb - ag}{ab - fc}$. Hinc æquatio erit $azdu + cudu = fzdz + budz$: quæ, quum in omnibus terminis habeat indeterminatas ad eamdem poteſtatem elevatas, pertinet ad Canonem Gabrielis Manfredii hominis doctiſſimi. Quare fiat $u = sz$ & poſt non ita multas operationes invenietur $\frac{-dz}{z} = \frac{ads + csds}{cs^2 + as - bs - f}$. Ex qua formula prorſus eadem conſectaria elicies, quæ ſupra commemoravimus.

XXII. Verum neque hæc methodus quidquam prodeſt, dum $ab = fc$. Si in hac hypoteſi foret etiam $ag = fb$, ex quibus duabus hypotheſibus tertia elicitur, nempe $bb = cg$, res eſſet

per

per se patens. Nam nostra æquatio rite disposita fieret

$axdy + bdy + cydy = \frac{f}{a} . \overline{axdx + bdx + cydx}$, quæ dividi potest per $ax + b + cy$, eaque divisa oritur $dy = \frac{fdx}{a}$, quæ integrata dat $A + y = \frac{fx}{a}$. Duæ itaque æquationes propositæ faciunt satis, nempe $ax + b + cy = o$, & $A + y = \frac{fx}{a}$.

XXIII. Verum quando in hypothesi $ab = fc$, quantum quidem scio, res nondum confecta est; ad separandas incognitas methodo usus sum non ita usitata, quæ me voti compotem effecit. Traditam æquationem hac ratione dispono

$x = \frac{y . \overline{bdx - cdy}}{ady - fdx} + \frac{gdx - bdy}{ady - fdx}$. Utor substitutione $x = S\, t\, dy$, & $dx = tdy$, & erit

$S\, tdy = \frac{y . \overline{btdy - cdy}}{ady - ftdy} + \frac{gtdy - bdy}{ady - ftdy}$, sive

$S\, tdy = \frac{y . \overline{bt - c}}{a - ft} + \frac{gt - b}{a - ft}$. Accommodemus formulam nostræ hypothesi, & cum sit $b = \frac{fc}{a}$, facta substitutione habebimus

$S\, tdy = \frac{y . \overline{fct - ac}}{a . \overline{a - ft}} + \frac{gt - b}{a - ft} = \frac{-cy}{a} + \frac{gt - b}{a - ft}$,

& differentiando, D dessignat differentialem quantitatis subsequentis,

$tdy + \frac{cdy}{a} = D\, \frac{gt - b}{a - ft}$ sive $dy = \frac{1}{t + \frac{c}{a}}\, D\, \frac{gt - b}{a - ft}$,

in qua inveniuntur incognitæ separatæ.

XXIV. Quantitas, quæ sita est sub signo D, differentietur & erit $dy = \frac{\overline{ga - fb}}{f} . \frac{dt}{\overline{t + \frac{c}{a}} . \overline{\frac{a}{f} - t}^2}$. Antequam progredior, adverto, quid eveniat in suppositione $\frac{-c}{a} = \frac{a}{f}$; tunc enim formula mutabitur in hanc

$dy = \frac{ga - fb}{f} \cdot \frac{-dt}{\overline{\frac{a}{f} - t}^3}$: Igitur integrando

$y = A + \frac{ga - fb}{f} \cdot \frac{-1}{2 \cdot \overline{\frac{a}{f} - t}^2}$ &

$tdy = dx = \frac{ga - fb}{f} \cdot \frac{-tdt}{\overline{\frac{a}{f} - t}^3}$, & integrando

$x = B + \frac{ga - fb}{f} \cdot \overline{\frac{-a}{2f \cdot \overline{\frac{a}{f} - t}^2} + \frac{1}{\frac{a}{f} - t}}$.

XXV. Inventi valores indeterminatarum x, y, qui algebraice dantur per t, ostendunt curvam esse algebraicam; imo facto calculo inveniemus esse sectionem conicam. Et quamquam in casu $\frac{ga - fb}{b} = o$ formulæ videantur deficere, tamen facto calculo reperies æquationem ad lineam rectam, quæ altera est ex supra traditis, ubi in hoc casu æquationem examinavimus N. XXII. Verum duæ hypotheses $b = \frac{fc}{a}$, $-\frac{c}{a} = \frac{a}{f}$ evidenter tertiam inferunt $a = -b$: qua in hypothesi semper propositam formulam integrabilem esse, supra monstravimus. N. XI.

XXVI. Quare hac hypothesi omissa universaliter formulam tractemus: quod ut facilius fiat, ponatur $\frac{ga - fb}{f} = m$, $\frac{c}{a} = p$ $\frac{a}{f} = q$; ita ut sit $dy = \frac{mdt}{t + p \cdot \overline{q - t}^2}$, quæ juxta notas methodos tractata exhibet

$dy =$

$$d\,y = \frac{m\,d\,t}{\overline{p+q}^2 \cdot \overline{p+t}} + \frac{m\,d\,t}{\overline{p+q}^2 \cdot \overline{q-t}} + \frac{m\,d\,t}{\overline{p+q} \cdot \overline{q-t}^2}\text{, quæ}$$

integretur, $y = l\mathrm{A} + l\,\overline{t+p} - l\,\overline{q-t} + \frac{m}{\overline{p+q} \cdot \overline{q-t}}$

ſumptis logarithmis in logiſtica, cujus ſubtangens $= \frac{m}{\overline{p+q}^2}$.

XXVII. Itaque ſi ope logarithmicæ deſcribamus curvam reſpondentem ultimæ æquationi, in qua t ſint ordinatæ, y abſciſſæ, ſpatium inter curvam, & coordinatas comprehenſum diviſum per unitatem $= x = S\,t\,d\,y$. Verum hoc ſpatium, atque adeo valor x per logarithmos invenietur. Eſt enim

$$t\,d\,y = d\,x = \frac{m\,t\,d\,t}{\overline{p+q}^2 \cdot \overline{t+p}} + \frac{m\,t\,d\,t}{\overline{p+q}^2 \cdot \overline{q-t}} + \frac{m\,t\,d\,t}{\overline{p+q} \cdot \overline{q-t}^2}$$

ſive

$$d\,x = \frac{m\,d\,t}{\overline{p+q}^2} - \frac{m\,p\,d\,t}{\overline{p+q}^2 \cdot \overline{t+p}} - \frac{m\,d\,t}{\overline{p+q}^2}$$
$$+ \frac{m\,q\,d\,t}{\overline{p+q}^2 \cdot \overline{q-t}} - \frac{m\,d\,t}{\overline{p+q} \cdot \overline{q-t}} + \frac{m\,q\,d\,t}{\overline{p+q} \cdot \overline{q-t}^2}.$$

Nam ſi ſummentur ſingula terminorum paria, ſuperior redibit æquatio. Ultima æquatione expurgata erit

$$d\,x = -\frac{m\,p\,d\,t}{\overline{p+q}^2 \cdot \overline{t+p}} - \frac{m\,p\,d\,t}{\overline{p+q}^2 \cdot \overline{q-t}} + \frac{m\,q\,d\,t}{\overline{p+q} \cdot \overline{q-t}^2},$$

quæ integrata ſumptis logarithmis in eadem, qua ſupra, logiſtica, fiet $x = -p\,l\,\overline{t+p} + p\,l\,\overline{q-t} + l\mathrm{B} + \frac{m\,q}{\overline{p+q} \cdot \overline{q-t}}$.

XXVIII. Hæc methodus ubique conſtructionem ſuppeditabit. Verum elengantius fortaſſe fiet, ſi vulgari methodo prætermiſſa, aliam ſequamur rei noſtræ accommodatiſſimam. Inventa jam eſt æquatio N. XXIII.

$d\,y =$

$dy = \frac{1}{t+\frac{c}{a}} D \frac{gt-b}{a-ft}$, ex qua oritur

$tdy = dx = \frac{t}{t+\frac{c}{a}} D \frac{gt-b}{a-ft}$. Fiat $\frac{gt-b}{a-ft} = z$,

$\frac{1}{t+\frac{c}{a}} = u$, & $\frac{t}{t+\frac{c}{a}} = W$. Igitur erit

$dy = udz$, & $dx = Wdz$.

XXIX. Nihil jam reliquum eſt, niſi ut videamus, ad quas quadraturas pertineant formulæ udz, Wdz. Ajo, utramque ad hyperbolam inter aſſymptotos pertinere. Etenim ſi recte utaris formulis ſuppoſitis N. ſuperiore, invenies $t = \frac{az-b}{g+fz}$, $t = \frac{a-cu}{au}$, demum $t = \frac{cW}{a-aW}$. Ergo $\frac{az+b}{g+fz} = \frac{a-cu}{au}$, $\frac{az+b}{g+fz} = \frac{cW}{a-aW}$. Porro utraque æquatio pertinet ad hyperbolam inter aſſymptotos. Hac autem ratione conſtruuntur.

XXX. Rectæ (*Fig.*) A a, Q q ſeſe ad angulum rectum interſecent in puncto C. Sumatur $CA = Ca = \frac{ag-bf}{a^2+fc}$. Excitentur $AB = ab = \frac{aa}{aa+fc}$: & inter aſſympotos Q q, A a deſcribantur hyperbolæ oppoſitæ tranſeuntes per puncta B, b. Sumatur $CD = \frac{ab+cg}{aa+fc}$, & $CE = \frac{af}{aa+fc}$, & per E agatur E G T, & F D G per D parallelæ aſſymptotis, erunt $GT = z$ $TS = u$: Ergo ſpatium $FGTS = Sudz = y$ addita ſi lubet, vel detracta aliqua conſtante.

XXXI. Tum sumatur $EK = AB = \frac{aa}{aa + fc}$, & per K agatur h K H parallela C A, quæ concurrat cum A B, a h in punctis, H, h: Normales rectæ K H excitentur $HI = hi = \frac{ac}{a^2 + fc}$, & per puncta I, i describantur inter assymptotos H h, Q q hyperbolæ oppositæ, quarum altera concurrit cum F G in L: erunt $GT = z$, & $TY = w$, & spatium $LGTY = S\,w\,dz = dx$: quibus inventis nihil jam difficile est, supposita hyperbolæ quadratura, requisitam curvam describere, cujus progressus pro varia coefficientium proportione varius evadet.

XXXII. Hoc genus constructionis videtur locum non habere ubi $aa + fc = o$, nam omnes hyperbolarum parametri infinitæ evadunt. Verum in hoc casu hyperbolæ transeunt in lineas rectas, ut apparet ex formulis harum curvarum descriptis N. XXIX, quæ in ea hypothesi transeunt in formulas linearum rectarum: sed hunc casum jam pertractavimus N. XXIV. Si $a = o$ nostra constructio deficiet quidem, sed advertendum est, tunc b fieri infinitam propter æquationem, quam supponimus, $b = \frac{fc}{a}$ Quod si non minus a, quam $c = o$, formula eadem methodo pertractata, ad absolutam separationem perduceretur. Quod si existeret tam $a = o$, quam $f = o$, aut etiam $c = o$, res nihil haberet difficultatis.

XXXIII. Dixi, existentibus non minus $a = o$, quam $c = o$, formulam eadem ratione pertractatam ad separationem perduci, & ad idem genus constructionis: quod quamquam facile est, tamen breviter calculum indicabo. Æquatio enim huic casui accommodata est hujusmodi $S\,t\,dy = \frac{-by}{f} \quad \frac{-gt+b}{ft}$:

Ergo differentiando $t\,dy + \frac{b\,dy}{f} = D\,\frac{-gt+b}{ft}$, sive

$$dy = \frac{f}{ft+b} \cdot D\,\frac{-gt+b}{ft}, \ \& \ t\,dy = dx = \frac{ft}{ft+b} D\,\frac{-gt+b}{ft}:$$

ad quas formulas quum sit æquatio perducta, nihil difficile erit

ope

ope duarum hyperbolarum ex nostra ratione constructionem absolvere.

XXXIV. Unus casus reliquus est, ad quem hujusmodi constructio nequaquam se extendit, quum scilicet $ag - bf = o$: sed hic casus conjunctus cum casu nostræ hypothesis $ab = fc$, qui trahunt tertium $bb = cg$, pertractatus est N. XXII.

XXXV. Si æquationem nostram hac ratione disposuissem

$$y = \frac{x \cdot \overline{ady - fdx}}{bdx - cdy} + \frac{bdy - gdx}{bdx - cdy},$$ & posuissem $y = \int t\,dx$,

eodem prorsus calculo ad separationem deveniretur. Casus autem, quo frustraretur methodus, est quum $bb - gc = o$, qui prorsus coincidit cum casu methodi superioris, qui descriptus est N. XXXIV.

XXXVI. Quamquam nostra hæc methodus maxima simplicitate, & elegantia ornata est in hypothesi $ba = fc$, cui illam accommodavimus, quæ hypothesis frustra alia ratione tentatur: tamen etiam extra hanc hypothesim separationem indeterminatarum obtinet; quod breviter indicare sufficiet Repeto itaque æquationem

$$x = \frac{y \cdot \overline{bdx - cdy}}{ady - fdx} + \frac{gdx - bdy}{ady - fdx}.$$ Fiat $x = \int t\,dy$ & erit

$$\int t\,dy = \frac{y \cdot \overline{bt - c}}{a - ft} + \frac{gt - b}{a - ft},$$ quæ differentiata dat

$$t\,dy = \frac{dy \cdot \overline{bt - c}}{a - ft} + y\,D\frac{bt - c}{a - ft} + D\frac{gt - b}{a - ft},$$ sive

$$\frac{dy}{y} = \frac{a - ft}{at - ft^2 - bt + c} \cdot D\frac{bt - c}{a - ft} + \frac{a - ft}{y \cdot \overline{at - ft^2 - bt + c}} \cdot D\frac{gt - b}{a - ft}$$

Fiat $$\frac{a - ft}{at - ft^2 - bt + c} \cdot D\frac{bt - c}{a - ft} = \frac{-ds}{s},$$ & s dabitur saltem transcendenter per t, & erit

$$\frac{dy}{y} + \frac{ds}{s} = \frac{a - ft}{y \cdot \overline{at - ft^2 - bt + c}} \cdot D\frac{gt - b}{a - ft}$$ Ponatur

$$\frac{dy}{y} + \frac{ds}{s} = \frac{du}{u},$$ & $ys = u$, & $y = \frac{u}{s}$, quibus substitutis

du

$$\frac{du}{u} = \frac{\overline{a - ft}.s}{u.\,\overline{at - ft^2 - bt + c}} . D\frac{gt - b}{a - ft} \text{ five}$$

$$du = \frac{\overline{a - ft}.s}{at - ft^2 - bt + c} . D\frac{gt - b}{a - ft}$$ cum incognitis feparatis.

XXXVII. Eadem methodus ad feparationem duxiffet, fi pofitum fuiffet $dy = S\,t\,dx$. Sed hæc fatis eft indicaffe. Quibus omnibus rite perpenfis apparebit, nullum effe cafum, in quo in propofita differentiali æquatione indeterminatæ feparari non poffint, imo non paucos effe cafus, qui pluribus rationibus abfolvuntur.

Additamentum primum.

XXXVIII. Methodus, qua ufus fum ad obtinendum indeterminatarum feparationem in ultimo cafu propofitæ formulæ, omnibus formulis accommodari poteft, quæ in hac æquatione œcumenica continentur $x = y\,M + a\,N$, in qua M, N dari fupponuntur per dx, dy, & conftantes. Primo quidem fi $M = o$, res ita liquido conftat, ut nullus fit dubitationi locus. Quare hoc omiffo id demonftrandum aggredior de formula omnibus terminis conftante. Quam ob rem hoc mihi præmittendum eft.

XXXIX. Si in formula $M\,dx + y\,N\,dx = dy$ dentur M, N quocumque modo per x, & per conftantes, indeterminatæ femper poterunt feparari. Nam dividatur æquatio per y, & fiet $\frac{M\,dx}{y} = \frac{dy}{y} - N\,dx$. Ponatur $N\,dx = \frac{dz}{z}$ atque adeo $S\,N\,dx = l\,z$; quare z data erit faltem tranfcendenter per x; Facta fubftitutione erit $\frac{M\,dx}{y} = \frac{dy}{y} - \frac{dz}{z}$. Ponatur $\frac{dy}{y} - \frac{dz}{z} = \frac{dt}{t}$, & erit $\frac{y}{z} = t$, & $y = zt$. Peractis iterum fubftitutionibus orietur $\frac{M\,dx}{zt} = \frac{dt}{t}$, five $\frac{M\,dx}{z} = dt$, in qua, quum z detur per x, incognitæ separatæ inveniuntur.

XL. Separata hac formula, quemadmodum alii Geometræ docuerunt, venio ad formulam $x = y\,M + a\,N$, in qua M, N datæ supponuntur per dx, dy, & constantes. Adverte oportere, ut quantitates M, N nullius sint dimensionis; si enim secus esset, lex homogeneorum non conservaretur: Ergo si pro dx substituam $t\,dy$, differentialia ex quantitatibus M, N omnino abibunt. Quapropter pono $x = S\,t\,dy$, & $dx = t\,dy$, & factis hisce substitutionibus dx, dy evanescent in quantitatibus M, N, & solum in eisdem inerit t. Quare æquatio hanc formam accipiet $S\,t\,dy = y\,P + a\,Q$, in qua P, Q dantur per t, & constantes. Sumantur itaque differentiæ $t\,dy = P\,dy + y\,dP + a\,dQ$, aut $dy = \frac{y\,.\,dP}{t - P} = \frac{a\,dQ}{t - P}$, in qua æquatione, quum adsint conditiones N. XXXIX., poterunt indeterminatæ separari. Quare licebit semper construere curvam, in qua y sint abscissæ, t ordinatæ. Atqui $x = S\,t\,dy$, idest æqualis spatio curvilineo curvæ modo descriptæ diviso per unitatem: igitur per hujusce quadraturam describetur etiam curva quæsita. Quæ quum ita sint liquido constat, æquationes, in quibus x, y unicam dumtaxat obtinent dimensionem, tametsi dx, dy quibuscumque afficiuntur exponentibus integris, aut fractis, separationem indeterminatarum recipere.

XLI. Unum aut alterum exemplum methodum declarabit, & aliquot non spernendis animadversionibus locum dabit. Data sit æquatio $x = \frac{y\,dx}{dy} + \frac{a\,dx^2}{dy^2}$. Fiat $x = S\,t\,dy$, & peractis necessariis substitutionibus fiet $S\,t\,dy = yt + at^2$, sumptisque differentiis $t\,dy = y\,dt + t\,dy + 2\,at\,dt$, quæ deletis terminis sese destruentibus exhibet $-y\,dt = 2\,at\,dt$. Quum hæc sit divisibilis per dt, dat æquationes duas, nimirum $dt = o$, $-y = 2\,a\,t$.

XLII. Si primam integremus habebimus $t = A$: Ergo $x = S\,A\,dy$, & integrando $x = Ay + B$, quæ pertinet ad lineam rectam. In hac formula duæ constantes inveniuntur, quæ additæ sunt in duabus integrationibus. Verum quum in æquatione, quæ construenda est, non adsint nisi differentialia primi ordinis

dinis, videtur unica dumtaxat constans locum habere posse; qua de re altera ex constantibus per alteram determinanda erit. Ut hoc fiat, accipiatur ultima formula $x = Ay + B$, differentietur $dx = A\,dy$. Hi valores collocentur in formula data

$$x = \frac{y\,dx}{dy} + \frac{a\,dx^2}{dy^2}$$, & erit $Ay + B = Ay + aA^2$. Hæc autem quum debeat esse idemtica, fiet $B = aA^2$. Quare vera integralis $x = Ay + aA^2$.

XLIII. Ad alteram æquationem venio, nempe $-y = 2at$, sive $t = \frac{-y}{2a}$: Ergo $x = S\frac{-y\,dy}{2a}$: Igitur integrando

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$x = A - \frac{y^2}{4a}$, quæ pertinet ad Parabolam Apollonianam. Ut determinetur valor constantis A, valores x, & dx substituantur in formula data, & invenietur $A - \frac{y^2}{4a} = \frac{-y^2}{2a} + \frac{ay^2}{4a^2} = \frac{-y^2}{4a}$; ex qua colligitur $A = o$: Ergo vera formula erit $x = \frac{-y^2}{4a}$.

XLIV. Si data æquatio methodo vulgari tractetur, hoc enim facere possumus, eædem integrales æquationes prodibunt. Namque adhibito opportuno calculo inveniemus

$dy = \frac{2a\,dx + y\,dy}{\sqrt{4ax + yy}}$. Fiat $\sqrt{4ax + yy} = z$, & erit $dy = dz$, & integrando $y + A = z = \sqrt{4ax + yy}$, & $yy + 2Ay + AA = 4aa + yy$: Ergo $x = \frac{2Ay + AA}{4a}$, quæ coincidit cum superiore. Altera æquatio nascitur, si fiat $z = \sqrt{4ax + yy} = o$. Ergo $4ax = -yy$.

XLV. Hæc proprietas duarum curvarum respondentium nostræ æquationi differentiali, ubique obtinetur, quum y non multiplicatur nisi per $\frac{dx}{dy}$; quia facto calculo secundum nostram me-

methodum, evanescet dy ex æquatione, quæ propterea erit divisibilis per dt; quemadmodum quisque experiundo comperiet.

XLVI. Exemplum alterum desumetur ab æquatione

$x = \frac{y\,dx^3}{dy^3} - \frac{a\,dx^2 - a\,dy^2}{dy^2}$. Pono $x = \int t\,dy$, & $dx = t\,dy$, factisque necessariis substitutionibus $\int t\,dy = yt^3 - at^2 - a$, & sumptis differentiis $t\,dy = t^3\,dy + 3yt^2\,dt - 2at\,dt$, sive $\overline{t - t^3}\,.\,dy = 3yt^2\,dt - 2at\,dt$, sive

$\frac{dy}{y} = \frac{3t\,dt}{1-tt}\ \frac{-2a\,dt}{y\,.\,\overline{1-tt}}$, aut $\frac{dy}{y} - \frac{3t\,dt}{1-tt} = \frac{-2a\,dt}{y\,.\,\overline{1-tt}}$.

Ponatur $\frac{dy}{y} - \frac{3t\,dt}{1-tt} = \frac{dz}{z}$, & integrando $ly + \frac{3}{2}l\,\overline{1-tt} = lz$, demum $y = \frac{z}{\overline{1-tt}^{\frac{3}{2}}}$, & peractis substitutionibus.

$\frac{dz}{z} = \frac{-2a\,dt\,.\,\overline{1-tt}^{\frac{3}{2}}}{z\,.\,\overline{1-tt}} = \frac{-2a\,dt\,.\,\sqrt{1-tt}}{z}$: Ergò

$\frac{dz}{2a} = -\,dt\,.\,\sqrt{1-tt}$: cujus constructio, ut notum est, dependet a circuli quadratura.

XLVII. Methodus hæc etiam in casibus aliis multis utilitati esse potest, ubi x solum linearem obtinet potestatem. Nam si fuerit $x = M$, in qua M detur quomodocumque per y, dx, dy, & constantes, posito $x = \int t\,dy$, & $dx = t\,dy$, factisque substitutionibus fiet $\int t\,dy = N$, in qua N dabitur per y, t, & constantes, neque in ea differentialia locum habebunt. Sumantur differentiæ, & erit $t\,dy = dN$, in qua dN dabitur per y, t, dy, dt & constantes; sed ita, ut differentialia linearem tantum obtineant dimensionem. Si in formula hac inventa methodus existit separandi indeterminatas, ut aliquando accidet, res erit perfecta; sin minus, nostra methodus quoque deficiet. Veruntamen per illam æquatio, in qua differentialia affecta sunt exponentibus quibuscumque vel integris, vel fractis, reducitur ad

æqua-

æquationem, in qua differentialia linearem tantum obtinent dimensionem.

XLVIII. Exemplum unicum propono. Sit æquatio

$$x = \frac{y\,dx}{dy} + \frac{ay^2\,dx^2}{dy^2} + \frac{by^3\,dx^3}{dy^3},$$ & ce. Facta $x = S\,t\,dy$ oritur

$S\,t\,dy = yt + ay^2t^2 + by^3t^3$ & ce., & differentiando $t\,dy = t\,dy + y\,dt + 2at^2y\,dy + 2ay^2t\,dt + 3bt^3y^2\,dy + 3by^3t^2\,dt$, & ce. sive $-y\,dt = 2at^2y\,dy + 2ay^2t\,dt + 3bt^3y^2\,dy + 3by^3t^2\,dt$ & ce. In qua quoniam licet separare indeterminatas, utilis est methodus. Fiat igitur $yt = z$, & $-y = \frac{-z}{t}$, & erit $\frac{-z\,dt}{t} = 2az\,dz + 3bz^2\,dz$ & ce. sive $\frac{-dt}{t} = 2a\,dz + 3bz\,dz$ & ce. Et integrando

$lA - lt = 2az + \frac{3}{2}bz^2$ & ce. In, qua quando habentur incognitæ separatæ, constructio est in potestate. Hæc dicta volo ut appareat, non omni methodum utilitate carere.

Additamentum alterum (a).

Cajetana Maria Agnesia Femina Clarissima in analyticis institutionibus æquationem $\overline{ax + b + cy}.\,dy = \overline{fx + g + hy}.\,dx$ pro casu, in quo $ab = fc$, methodo longe diversa absolvit. Quæ methodus propter elegantiam suam digna visa est, quæ hoc loco diligentius exponatur, ne quid pertinens ad nostram æquationem desideretur. Substitutio hæc usurpanda est $y = Az + Bx$, & facta differentiatione $dy = A\,dz + B\,dx$, in quibus A, B sunt quantitates constantes determinandæ in operationis progressu. Peractis substitutionibus æquatio hanc formam induit

$\overline{a + cB}$.

(a) *Nunc primum in lucem prodit simul cum aliis duobus, quæ sequuntur additamenta.*

$\overline{a+cB}.Axdz+\overline{a+cB}.Bxdx+bAdz+bBdx+cA^2zdz$
$+cABzdx=\overline{f+bB}.xdx+gdx+bAzdx.$

Ut æquatio fiat brevior per congruam determinationem speciei B eliminetur terminus xdz: quam ob rem necesse est, ut $B=-\frac{a}{c}$. Hoc autem valore introducto oritur æquatio

$bAdz-\frac{ba}{c}dx+cA^2zdz-aAzdx=\overline{f-\frac{ab}{c}}.xdx$
$+gdx+bAzdx$. Quoniam vero in nostra hypothesi $fc=ab$, erit $f-\frac{ab}{c}=o$: Ergo terminus xdx abibit ab æquatione, quæ proinde formam hanc accipiet translatis terminis $bAdz+cA^2zdz$
$=\overline{g+\frac{ba}{c}}.dx+A.\overline{a+b}.zdx$ sive $\frac{bAdz+cA^2zdz}{g+\frac{ba}{c}+A.\overline{a+b}.z}=dx$,

in qua indeterminatæ existunt separatæ. Quantitas A pro libito determinari poterit, dummodo non fiat $=o$.

Simili ratione si utaris substitutione $x=Az+By$, & arceas x ab æquatione, pro eadem hypothesi $fc=ab$ æquationem a permixtione indeterminatarum liberabis.

Additamentum tertium.

Blandior mihimetipsi non mediocriter, quod methodus separandi indeterminatas in formula $x=yM+N$, in qua M, N datæ supponuntur per dx, dy, & constantes, quam in additamento typis mandavi anno 1747, adeo probata fuerit Allemberto Analystæ acutissimo, ut eam primum exposuerit in tertia parte Disquisitionum in Calculum Integralem, quæ edita est in tomo quarto actorum Berolinensium anno 1750. Juvabit hic in latinum ex gallico sermone convertere, quæ scribit Analysta maximus, ut Lector facilius non injucundam comparationem possit instituere. Accipe Allemberti verba.

„ Supponam ubique, quæ sequuntur, propositiones „ $z = \frac{dx}{dy}$, $u = \frac{dz}{dy}$ aut $\frac{ddx}{dy^2}$, $k = \frac{du}{dy}$ aut $\frac{dddx}{dy^3}$ & ce.

PROBLEMA.

„ Invenire integrale æquationis differentialis, quæ continet „ quascumque functiones elementorum dx, dy, & in qua inve- „ niuntur x, y, dummodo neqne inter sese multiplicentur, ne- „ que altera alteram dividat, neque eleventur ad potestatem uni- „ tate majorem.

„ Patens est, hæc æquationum genera posse semper repræ- „ sentari per formulam $x = y \varphi z + \Delta z$, φz, & Δz exprimentibus „ functiones quascumque z, seu $\frac{dx}{dy}$. Differentietur hæc formu- „ la, & pro dx scribatur ejus valor $z\,dy$, & habebitur „ $z\,dy = dy \cdot \varphi z + y\,d(\varphi z) + d(\Delta z)$: ex qua æquatione „ facile colligetur valor y in z, atque adeo valor x, quum sit „ $x = S\,z\,dy$.

COROLLARIUM PRIMUM.

„ Eodem modo probabitur, integrari posse æquationem, quæ ad „ hanc formam reducitur $z = y \varphi u + \Delta u$, aut $u = y \varphi k + \Delta k$ & ce. „ ex quo consequitur, generatim omnem æquationem, quæ in- „ cludit y, & $d^n x$ lineares, & functiones quascumque $d^{n+1} x$, & „ dy, integrationem accipere ex methodo præsentis proble- „ matis.

COROLLARIUM SECUNDUM.

„ Si $x = y \varphi z$, habetur casus multos ante annos cognitus „ æquationis homogeneæ.

„ Si $x = yz + \Delta z$, æquatio pertinet simul & ad lineam re- „ ctam, & ad curvam. Nam differentiatio dat $y + \Gamma z \cdot dz = o$;

ex

„ ex qua colligitur $dz = o$, quæ pertinet ad lineam rectam,
„ aut $y = - \Gamma z$, quæ est ad lineam curvam.

„ Observare licet, æquationem $dx + \frac{ax + by + c}{gx + hy + f} dy = o$,
„ de qua plures geometræ egerunt, esse casum peculiarem nostri pro-
„ blematis.

Hactenus Allembertus. Si comparationem instituas, cognosces, nihil a me prætermissum esse ex illis, quæ hic adnotata sunt, si primum corollarium excipias. In hoc continetur exigua quædam, sed non contemnenda hujusce nostræ methodi accessio, quam acceptam referimus Allemberto. Non ingratum erit si hanc quoque fusius, & appositis exemplis modo nostro explicemus.

Sit tractanda æquatio $d^n x = yM + N$, in qua, sumpto constante elemento dy, M, N datæ supponuntur per dy, $d^{n+1}x$, & constantes. Sit $d^{n+1}x = t\, dy^{n+1}$, & integrando $d^n x = dy^n S t\, dy$. Facta substitutione æquatio accipiet hanc formam $dy^n S t\, dy = y\, dy^n P + dy^n Q$, quæ dividi potest per dy^n. Quantitates P, Q datæ invenientur per t, & constantes. Igitur æquatio fiet $S t\, dy = y P + Q$, quæ differentiata exhibet $t\, dy = P\, dy + y\, dP + dQ$, sive $t - P \,.\, dy = y\, dP + dQ$, in qua semper indeterminatæ separari possunt, & inveniri y per t, aut vice versa.

Ad inveniendam x integretur æquatio $d^n x = dy^n S t\, dy$, & fiet $d^{n-1} x = dy^{n-1} \overline{S dy S t dy}$: facta nova integratione $d^{n-2} x = dy^{n-2} S dy \overline{S dy \overline{S t dy}}$; atque ita deinceps donec pervenias ad $d^{n-n} x = x = \overline{S dy S dy \ldots\ldots S t dy}$, in qua numerus signorum S erit $n+1$. Adverte, in accipiendis summatoriis additionem constantis non esse omittendum. Sed de hac additione paullo infra monebo nonnihil. Nunc aliquot proponamus exempla, quæ utilibus animadversionibus locum præbebunt.

Exem-

Exemplum primum. Sit proposita æquatio $ddx = \frac{yd^3x}{dy} + \frac{a^3dy^5}{d^3x}$. Utere substitutione $d^3x = tdy^3$, & integrando $d^2x = dy^2 S tdy$, ex quibus resultat æquatio $dy^2 S tdy = ytdy^2 + \frac{a^3dy^2}{t}$, sive facta divisione per dy^2, $S tdy = yt + \frac{a^3}{t}$, & differentiis acceptis $tdy = tdy + ydt - \frac{a^3dt}{t^2}$, sive $y - \frac{a^3}{t^2} . dt = o$.

Ex hac duplex oritur æquatio, nempe $y - \frac{a^3}{t^2} = o$, sive $t = \frac{a\sqrt{a}}{\sqrt{y}} = \frac{d^3x}{dy^3}$, sive $2a\sqrt{a} . dy^2 . \frac{dy}{2\sqrt{y}} = d^3x$, & integrando $2a\sqrt{a} . \sqrt{y} . dy^2 + Ady^2 = d^2x$, iterum integrando $\frac{2 . 2a\sqrt{a} . y^{\frac{3}{2}}dy}{3} + Aydy + Bdy = dx$, novaque facta integratione $\frac{2 . 2 . 2a\sqrt{a} . y^{\frac{5}{2}}}{3 . 5} + \frac{Ay^2}{2} + By + C = x$. Valores d^2x, d^3x inventi in propositam æquationem introducantur, & fiet $2a\sqrt{a} . \sqrt{y} . dy^2 + Ady^2 = \frac{a\sqrt{a} . ydy^2}{\sqrt{y}} + \frac{a^3dy^2 . \sqrt{y}}{2a\sqrt{a}}$, quæ expurgata sufficiet $2a\sqrt{a} . \sqrt{y} + A = 2a\sqrt{a} . \sqrt{y}$, quæ quum debeat esse idemtica, manifesto demonstrat $A = o$. Itaque ve-

ra

ra æquatio erit $\frac{2.2.2 a\sqrt{a}.y^{\frac{5}{2}}}{3.5} + By + C = x.$

Altera, quæ oritur æquatio, est hujusmodi $dt = o$, & integrando $t = A = \frac{d^3 x}{dy^3}$, sive $A dy^3 = d^3 x$, & integrando $Aydy^2 + Bdy^2 = d^2 x$; integretur denuo $\frac{Ay^2 dy}{2} + Bydy + Cdy = dx$: demum $\frac{Ay^3}{2.3} + \frac{By^2}{2} + Cy + E = x$: quæ est generis parabolici.

Hoc autem semper constans est, ut quum alter terminus æquationis $= \frac{yd^{n+1}x}{dy}$ semper provenit $dt = o$, quæ curvam præbet generis parabolici. Multiplicator autem elementi dt dabit æquationem inter z, & y sine integratione.

In ultima æquatione quatuor constantes existunt, quum æquatio tertio-differentialis tres tantum admittat. Una igitur per alias determinanda erit. Ut hoc fiat, inventi valores ddx, $d^3 x$ substituantur in data æquatione, & proveniet $Aydy^2 + Bdy^2 = \frac{Aydy^3}{dy} + \frac{a^3 dy^5}{A dy^3}$, sive $Ay + B = Ay + \frac{a^3}{A}$, quæ quum debeat esse idemtica, necessario $B = \frac{a^3}{A}$. Quare genuina æquatio fiet $\frac{Ay^3}{2.3} + \frac{a^3 y^2}{2A} + Cy + E = x$. Q.E.I.

Exemplum alterum. Proposita sit æquatio $ad^3 x = \frac{yd^4 x^2}{dy^5} + bdy^3$. Pone $d^4 x = tdy^4$, & $d^3 x = dy^3 S tdy$, factaque substitutione orietur $ady^3 S tdy = yt^2 dy^3 + bdy^3$. Divide per dy^3,

tum

tum sume differentias $atdy = t^2dy + 2ytdt$, sive $\frac{dy}{y} = \frac{2dt}{a-t}$. Integra, & habebis $ly = lA - 2l\overline{a-t}$, & facto transitu ad numeros $a - t = \frac{\sqrt{A}}{\sqrt{y}}$, aut $a - \frac{\sqrt{A}}{\sqrt{y}} = t$.

Itaque $d^4x = d^3 . \overline{ady - \frac{dy\sqrt{A}}{\sqrt{y}}}$. Integra $d^3x = dy^3 . \overline{B + ay + 2\sqrt{A} . \sqrt{y}}$. Antequam progredior, pono in æquatione proposita valores d^4x, d^3x, ut determinem, si opus est, quantitates B, A altera per alteram. Fit autem

$ady^3 . \overline{B + ay - 2\sqrt{A}\sqrt{y}} = ydy^3 . \overline{a^2 - \frac{2a\sqrt{A}}{\sqrt{y}} + \frac{A}{y} + \frac{b}{y}}$, sive $aB + a^2 - 2a\sqrt{A}\sqrt{y} = a^2y - 2a\sqrt{A}\sqrt{y} + A + b$, quæ æquatio vera esse non potest, nisi sit $aB = A + b$, seu $B = \frac{A+b}{a}$.

Determinato jam valore B per A iterentur integrationes

$$d^2x = dy^2 . \overline{C + By + \frac{ay^2}{2} - \frac{2 . 2\sqrt{A} . y^{\frac{3}{2}}}{3}},$$

$$dx = dy . \overline{E + \frac{By^2}{2} + \frac{ay^3}{2 . 3} - \frac{2 . 2 . 2\sqrt{A}y^{\frac{5}{2}}}{3 . 5}},$$

$$x = F + Ey + \frac{Cy^2}{2} + \frac{By^3}{2 . 3} + \frac{ay^4}{2 . 3 . 4} - \frac{2 . 2 . 2 . 2\sqrt{A}y^{\frac{7}{2}}}{3 . 5 . 7},$$

quæ æquatio, substituto pro B ejus valore supra determinato, exhaurit propositam æquationem differentialem.

Exemplum tertium. Tractanda sit æquatio

$2\,d^3x = -y\cdot\overline{\frac{a^2dy^7}{d^4x} - \frac{d^4x}{dy}} + \frac{a\,d^4x}{dy}$. Fac $d^4x = t\,dy^4$, &

$d^3x = \underline{dy^3\,S\,t\,dy}$, quibus ſubſtitutis habetur $2\,\underline{dy^3\,S\,t}\,dy =$
$-y\cdot\frac{a^2dy^3}{t} - t\,dy^3 + a\,t\,dy^3$, ſive $2\,S\,t\,dy = -y\cdot\underline{\frac{a^2}{t} - t} + a^2t$,

& differentiis acceptis $2\,t\,dy = -dy\cdot\frac{a^2}{t} - t$
$+ y\cdot\frac{a^2dt}{t^2} + dt + a\,dt$, & terminis tranſpoſitis

$dy\cdot\overline{\frac{a^2}{t} + t} = y\cdot\frac{a^2dt}{t^2} + \overline{dt + a\,dt}$, & neceſſariis peractis

operationibus $\frac{dy}{y} - \frac{dt}{t} = \frac{a\,t\,dt}{y\cdot\overline{a^2 + t^2}}$. Fiat $\frac{dy}{y} - \frac{dt}{t} = \frac{dr}{r}$,
ex qua colligatur $a\,y = t\,r$. Subſtituendo invenies
$\frac{dr}{r} = \frac{a\,t\,dt}{t\,r\cdot\overline{aa + tt}}$, ſive $dr = \frac{a^2dt}{aa + tt}$, quæ ſpectat ad circuli quadraturam.

Juvabit definire, quænam conſtans addenda ſit in ſumenda $S\,t\,dy$, ſi ultima formula ita conſtruatur, ut, facta $t = o$, etiam ſit $r = o$. Conſtat, in ſuppoſitione $t = o$, fore

$dt = dr$, $y = o$, & $dy = \frac{dt^2}{a}$. His ſtatutis ſupponamus in hac hypotheſi $S\,t\,dy = B$, quo valore in ſuperiore formula ſubſtituto fiet $2\,B = \frac{-a^2y}{t}$, quæ præbet fractionem $\frac{o}{o}$. Quapropter

pter ex mea methodo pro y, t, colloca $o + dy$, $o + dt$, & obtinebis $2\,B = \frac{-a^2 dy}{dt}$: atqui $dy = \frac{dt^2}{a}$: ergo $2\,B = -a\,dt$, nempe B infinitesima erit, seu nulla. Quare ita sumenda erit $S\,t\,dy$, ut, nullescentibus t, y, ipsa quoque nullescat.

Quod si, facta $t = o$, sit $r = A$, erit quidem $dt = dr$, $y = o$, sed $dy = \frac{A\,dt}{a}$: ergo $2\,B = -A\,a$. Quare in hac hypothesi ita accipienda erit $S\,t\,dy$, ut, evanescentibus, t, , ipsa fiat $= \frac{-A\,a}{2}$.

De hac determinatione constantis nihil dixit Allembertus. Verum, quemadmodum olim quoque in additamento notavi, ita necessaria est, ut nisi fiat, quæ invenitur æquatio, minime satisfaciat datæ æquationi differentiali.

Additamentum quartum.

Æquatio, in qua $d^n x$ invenitur dimensionis tantum linearis, neque aliæ quantitates habentur præter $d^{n+1} x$, y, dy quocumque modo inter se, & cum constantibus permixtæ, & ad quamcumque potestatem elevatæ, semper reducitur ad æquationem primo-differentialem. Res est per sese patens. Nam posito $d^{n+1} x = t\,dy^{n+1}$, & $d^n x = dy^n S\,t\,dy$, factisque substitutionibus, dy per divisionem ab æquatione abibit, & solum remanebit $S\,t\,dy$ dimensionis linearis cum y, & t. Quare congrua facta præparatione, & differentiatione prodibit æquatio primo-differentialis, per quam si inveniatur t data per y, jam x per y sine dubio determinabitur.

Multis modis evenire potest, ut æquatio differentialis primi ordinis, ad quam pervenimus, resolvi possit per indeterminatarum separationem, ut in præsens notum est geometris. Aliquot exempla in medium feram, ut appareat, methodum hanc non adeo angustis finibus coarctari.

Exem-

Exemplum primum ſuppeditabit æquatio $d^n x = \frac{a y^p d^{n+1} x^{2-p}}{d y^{n+2-pn-p}} + \frac{b d^{n+1} x^2}{d y^{n+2}}$. De more ſtatuo $d^{n+1} x = t d y^{n+1}$, & facta ſubſtitutione, & diviſione per $d y^n$ exurgit $S t d y = a y^p t^{2-p} + b t^2$. Sumatur differentia $t d y = p a t^{2-p} y^{p-1} d y + \overline{2-p}. a y^p t^{1-p} d t + 2 b t d t$. Hæc æquatio, quum pertineat ad canonem Gabrielis Manfredii viri doctiſſimi, a permixtione indeterminatarum liberatur. Adverte, plures exiſtere poſſe terminos, in quibus locum habeat y, & quidem elevata ad diverſas poteſtates, dummodo exponens $d^{n+1} x$ ſit æquale binario dempto exponente y.

Exemplum alterum exhibeat æquationem affectam ſigno radicali, quæ ſit hujuſmodi $d^n x = \sqrt{\frac{y^4 d y^{2n} + a d^{n+1} x^4}{d y^{2n+4}}}$. Subſtitutio $d^{n+1} x = t d y^{n+1}$ mutabit æquationem hoc modo $d y^n S t d y = \sqrt{y^4 d y^{2n} + a t^4 d y^{2n}} = d y^n \sqrt{y^4 + a t^4}$, ſive $S t d y = \sqrt{y^4 + a t^4}$, & differentiando

$$t d y = \frac{2 y^3 d y + 2 a t^3 d t}{\sqrt{y^4 + a t^4}},$$

in qua ſumma exponentium indeterminatarum in omnibus terminis eadem eſt, atque adeo ex nota methodo indeterminatæ ſeparantur.

Exemplum tertium ſimile eſt illi, quod extremo loco in primo additamento propoſui. Sit æquatio $d^n x = \frac{a y d^{n+1} x}{d y} + \frac{b y^2 d^{n+1} x^2}{d y^{n+2}} + \frac{c y^3 d^{n+1} x^3}{d y^{2n+3}}$ &c. Fac de more

$d^{n+1} x$

$d^{n+1}x = t\,dy^{n+1}$, tum substitue, & divide per dy^n, $S\,t\,dy = ayt + by^2t^2 + cy^3t^3$ & ce., & differentiando $t\,dy = at\,dy + ay\,dt + b\,Dy^2t^2 + c\,Dy^3t^3$ & ce., vel $\overline{1-a}.\,t\,dy - ay\,dt = b\,Dy^2t^2 + c\,Dy^3t^3$ & ce. Pone $yt = r$, & $y = \frac{r}{t}$, & $dy = \frac{t\,dr - r\,dt}{t^2}$, & invenies $\overline{1-a}.\,\frac{t\,dr - r\,dt}{t} - \frac{ar\,dt}{t} = 2br\,dr + 3cr^2dr$ & ce., sive $\frac{-r\,dt}{t} = \overline{a-1}.\,dr + 2br\,dr + 3cr^2dr$ & ce. Demum $\frac{-dt}{t} = \overline{a-1}.\,\frac{dr}{r} + 2b\,dr + 3cr\,dr$ & ce., in qua incognitæ separatæ inveniuntur.

Hæc, quæ paucis attacta sunt, patefaciunt, methodum hanc, tametsi y linearem tantum non obtineat dimensionem, minime utilitate carere. Veruntamen etiamsi æquatio primo-differentialis, ad quam pervenimus, indeterminatarum separationem respuat, tamen vel maxime utile erit, ad hanc reducere æquationem differentialem ordinis superioris. Etenim quum omnes æquationes primo differentiales, ut demonstravi in commentario *De usu motus tractorii in constructione æquationum differentialium*, per motum tractorium construi possint, & curva describi ipsis satisfaciens, ope hujusce curvæ construetur æquatio proposita differentialis, ad quemcumque differentialium ordinem pertineat.

Si æquatio præter $d^n x$ linearem contineat $d^{n+1}x$, $d^{n+2}x$ simul cum y, dy, quod elementum constans est, per eamdem substitutionem semper redigetur ad æquationem differentialem secundi ordinis. Hoc vix eget demonstratione: nam posita $d^{n+1}x = t\,dy^{n+1}$, factaque substitutione, & divisione per dy^n, invenietur $S\,t\,dy$ æqualis quantitati compositæ ex y, dy, t, dt: quare acceptis differentiis exurget æquatio differentialis secundi ordinis.

Exem-

Exemplum primum. Resolvenda sit æquatio $d^n x = \frac{y\, d^{n+1} x}{dy} + \frac{a\, d^{n+2} x}{dy^2}$. Substitutione consueta utere $d^{n+1} x = t\, dy^{n+1}$, quæ integrata præbet $d^n x = dy^n\, S\, t\, dy$, differentiata vero $d^{n+2} x = dt\, dy^{n+1}$. Peractis substitutionibus, & divisione per dy^n orietur $S\, t\, dy = y\, t + \frac{a\, dt}{dy}$. Differentiæ accipiantur $t\, dy = t\, dy + y\, dt + \frac{a\, ddt}{dy}$, sive $y\, dt = \frac{-a\, ddt}{dy}$: æquatio differentialis secundi ordinis, quæ nullo negotio completam resolutionem recipit. Nam ita disponatur $y\, dy = \frac{-a\, ddt}{dt}$, tum integretur $\frac{y^2}{2} = l\, A\, dy - l\, dt$, sive $c^{\frac{y^2}{2}} = \frac{A\, dy}{dt}$: c est numerus, cujus logarithmus æquat unitatem: ergo $dt = A\, c^{\frac{-y^2}{2}}\, dy$, cujus constructio est in potestate.

Exemplum alterum. Si proposita fuisset æquatio $d^n x = \frac{y\, d^{n+1} x}{dy} + \frac{a\, d^{n+2} x^m}{dy^{m-1 \,.\, n+2m}}$, institutis iisdem operationibus ad hanc æquationem differentialem secundi ordinis devenisses $-y\, dt = \frac{m\, a\, dt^{m-1}\, ddt}{dy^m}$, sive $-y\, dy^m = m\, a\, dt^{m-2}\, ddt$: quæ æquatio differentialis secundi ordinis ulteriorem resolutionem admittit. Nam integratà dat $\frac{A\, dy^{m-1} - y^2\, dy^{m-1}}{2} = \frac{m\, a\, dt^{m-1}}{m-1}$,

sive

five $dy.\overline{A - y^2}^{\frac{1}{m-1}} = \left(\frac{2ma}{m-1}\right)^{\frac{1}{m-1}} dt$, in qua indeterminatæ separatæ inveniuntur.

Exemplum tertium proponit resolvendam æquationem $d^n x = \frac{a d^{n+1} x}{dy} + \frac{b y d^{n+2} x}{dy^2}$, quæ factis de more substitutionibus, & divisione per dy^n in hanc mutatur $S t dy = at + \frac{b y dt}{dy}$, quæ differentiata præbet æquationem secundo differentialem, nempe $t dy = a + b . dt + \frac{b y d d t}{d y}$. Quoniam hæc æquatio continet t, aut ejus differentiale ad eamdem potestatem elatam, methodo Euleri viri doctissimi resolvetur. Suppone itaque $t = e^{S r dy}$, & $dt = e^{S r dy} r dy$, & $ddt = e^{S r dy} r^2 dy^2 + e^{S r dy} dr dy$. Effectis substitutionibus invenies $e^{S r dy} dy = \overline{a + b} e^{S r dy} r dy + b e^{S r dy} r^2 y dy + b e^{S r dy} y dr$, quæ divisa per $e^{S r dy}$ sufficit æquationem primo-differentialem, nempe $dy = \overline{a + b} r dy + b r^2 y dy + b y dr$, in qua quisque novit separare indeterminatas.

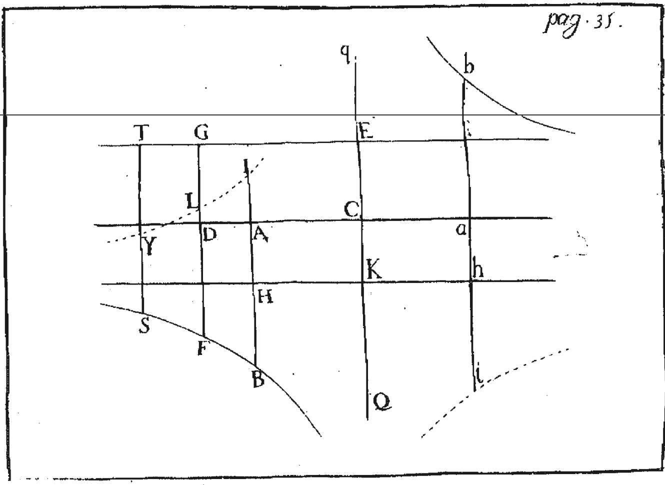
Hæc eadem resolutio locum habet, si in æquatione pro y collocata fuisset quæcunque functio ejusdem y. Sit hæc functio M, cujus differentialis ponatur esse N dy. Itaque æquatio $d^n x = \frac{a d^{n+1} x}{d y} + \frac{b M d^{n+2} x}{dy^2}$, factis iisdem substitutionibus, in hanc mutabitur $S t dy = at + \frac{b M d t}{d y}$, quæ differentiata exhibet $t dy = a dt + b N dt + \frac{b M d d t}{d y}$. Hæc æquatio secundo differentialis ad primas differentias redigetur per substitutionem $t = e^{S r dy}$: fiet enim $e^{S r dy} d y = a e^{S r dy} r d y$
$+ b N e$

$+ bNe^{Srdy} r dy + bMe^{Srdy} r^2 dy + bMe^{Srdy} dr$, & facta divisione per e^{Srdy}, $dy = ar\,dy + bNr\,dy + bMr^2 dy + bM\,dr$, quæ saltem per motum tractorium constructionem accipiet.

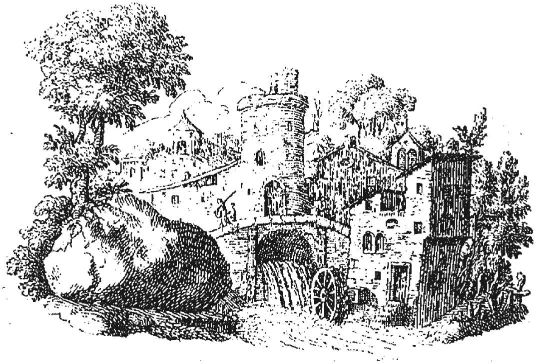
Liceat addere exemplum quartum, quod sua se simplicitate commendat. Proponatur itaque æquatio $d^n x = \frac{d^{n+1} x^p . d^{n+2} x^q}{dy^{p+q-1.n+p+2q}}$, quæ per consuetam substitutionem in hanc convertetur $St\,dy = \frac{t^p dt^q}{dy^q}$. Acceptis differentiis orietur æquatio secundo-differentialis, nimirum $t\,dy = \frac{pt^{p-1} dt^{q+1} + qt^p dt^{q-1} ddt}{dy^q}$. Ut ad primas differentias redigatur, per methodum dimidiatæ separationis ita disponatur æquatio $dy^{q+1} = t^{p-1} dt^q . \frac{pdt}{t} + \frac{qddt}{dt}$. Fiat $\frac{pdt}{t} + \frac{qddt}{dt} = \frac{dr}{r}$, ex qua oritur $t^p dt^q = r\,dy^q$ sive $\frac{t^{\frac{p}{q}} dt}{r^{\frac{1}{q}}} = dy$. Itaque peractis substitutionibus habebimus

$$\frac{t^{\frac{q+p}{q}} . dt^{q+1}}{r^{\frac{q-1}{q}}} = \frac{t^{p+1} dt^q dr}{r}$$, sive $t^{\frac{p+q}{q}} dt = r^{\frac{1}{q}} dr$: quæ integrabilis est aut absolute, aut per logarithmos, si vel $q = -1$, vel $p = -2q$.

Eadem prorsus facta substitutione, si æquatio præter $d^n x$ linearem contineat $d^{n+1} x$, $d^{n+2} x$, $d^{n+3} x$ simul cum y, & dy, redigetur ad æquationem tertio-differentialem. Si addatur



tur $d^{n+4}x$, æquatio orietur quarto differentialis: atque ita deinceps. Quare regula hæc latissime patet. Non sum nescius, plerumque desiderari methodos ad resolvendas æquationes secundo, tertio, quarto-differentiales. Verum hoc præsenti methodo non est vitio tribuendum, sed dolendum, quod calculus integralis longe adhuc absit a perfectione.



OPUSCULUM SECUNDUM

De Sectionum Conicarum rectificatione, ejusque usu.

E P I S T O L A

In qua determinantur arcus sectionum conicarum, quorum differentia rectificabilis est.

VINCENTIUS RICCATUS

JACOBO MARISCOTTO

In Bononiensi Instituto Geographiæ, & Nauticæ Professori

S. P. D.

Nullus dubito, Jacobe Ornatissime, quin ad plenam sectionum conicarum tractationem, in qua non minus proprietates illæ patefiant, quæ demonstrantur per geometriam finitorum, quam illæ, quæ sublimi indigent geometria infinitesimorum, opus sit, de earum rectificatione distincte verba facere. Videbitur fortasse res hæc non longam orationem desiderare. Nam si probemus, rectificationem circuli esse cum ejus quadratura conjunctam, parabolæ rectificationem a quadratura hyperbolæ dependere, ellypsis autem, & hyperbolæ rectificationem esse sui generis, neque obtineri posita sectionum conicarum quadratura; si demum exhibeamus series aliquas convergentes, per quas curvarum mensuram proximam determinemus; videmur, omnium votis satisfecisse, neque apparet, quidquam, quod addi possit, superesse.

Nihilo tamen minus Comes Julius Carolus de Fagnanis Analysta ingeniosus in vigesimo quarto, & in vigesimo sexto T. diarii Licteratorum Italiæ demonstravit, in omnibus coni sectionibus parabola, ellypsi, hyperbola existere arcus, quorum differentia est algebraice rectificabilis. Tanta est harum proprie-

tatum

tatum nobilitas, ut nullo pacto silentio premendæ sint. Verum quoniam Doctissimus Auctor easdem deducit ex theorematibus quibusdam universalibus, quæ licet in altioribus curvis non exiguum præstent usum, tamen demonstrationes sufficiunt non ita simplices, atque elegantes; rem tibi gratam me facturum confido, si in eo operam posuero, ut non modo easdem proprietates simplicius demonstrem, sed etiam aliquot veritates addam præsertim in parabola, quæ virum ingeniosissimum effugerunt.

Quandoquidem rectificatio parabolæ (ab hac enim curva exordium ducam) dependet, ut notum est ab hyberbolæ quadratura, hac utar, ut ostendam, quinam sint arcus parabolici, quorum differentia rectificationem recipit algebraicam.

Sit parabola 2 F C P, quæ tangatur in præcipuo (*Fig. I.*) vertice C a recta 2 B M. Acceptis abscissis C H $= x$, & ordinatis H I $= y$, sit æquatio curvæ $xx = 2ay$: igitur sumptis differentiis $xdx = ady$. Hinc elemento curvæ vocato $= ds$, habebitur $ds^2 = \frac{x^2 dx^2}{a^2} + dx^2$, aut $a^2 ds^2 = dx^2 . \overline{a^2 + x^2}$, extractaque radice $ads = dx \sqrt{a^2 + x^2}$. Quæ formula indicat, rectificationem parabolæ dependere a quadratura hyperbolæ.

Rectæ C H normalis agatur C A $= a$, & centro C, vertice A describatur hyperbola æquilatera 2 E A K. Producatur I H in K. Rectangulum semiparametri a, & curvæ C I æquat spatium hyperbolicum C A K H. Similiter ducta alia ordinata P M N fiet a. C P $=$ C A N M. Igitur deducta prima æquatione ab altera, reliqua erit a. I P $=$ H K N M. Quapropter quilibet arcus parabolicus ductus in semiparametrum æquabit spatium hyperbolicum clausum inter easdem ordinatas.

Per punctum C ducatur hyperbolæ assymptotum C T faciens cum axe C A angulum semirectum. Acceptis C G, C L, C Q continue proportionalibus, ductisque ordinatis assymptoto G E, L K, Q N, scimus, spatia hyperbolica G E K L, L K N Q æqualia esse. Ex punctis E, K, N agantur ordinatæ E B F, K H I, N M P. Spatium hyperbolicum

H K N M $=$ H S T M $+$ S L K $+$ L K N Q $-$ T N Q

B E K H $=$ B O S H $+$ O E G $+$ G E K L $-$ S K L

igi-

igitur

$a.\overline{IP} = HSTM + SLK + LKNQ - TNQ$

$a.\overline{FI} = BOSH + OEG + GEKL - SKL.$

Si prima ex his æquationibus ab altera detrahatur, deletis spatiis illis hyperbolicis, quæ propter æqualitatem eliduntur, remanebit

$a.\overline{IP} - \overline{FI} = HSTM - BOSH + 2.SLK - OEG - TNQ$, in quam quum solæ figuræ rectilineæ ingrediantur, constat, differentiam arcuum IP, FI esse algebraice rectificabilem.

Ut trapetia ad triangula redigantur, advertendum est,

$HSTM = CMT - CHS,$

$BOSH = CHS - CBO$: ergo

$$a.\overline{IP} - \overline{FI} = CMT - 2.CHS + CBO - TNQ + 2.SLK - OEG.$$

Si speciebus analyticis utaris, vocatis $CB = BO = b$, $CH = HS = x$, $CM = MT = z$, invenies

$$CG = \frac{b + \sqrt{aa+bb}}{\sqrt{2}},\quad CL = \frac{x + \sqrt{aa+xx}}{\sqrt{2}},$$

$$CQ = \frac{z + \sqrt{aa+zz}}{\sqrt{2}}.$$ Præterea

$$GE = GO = \frac{-b + \sqrt{aa+bb}}{\sqrt{2}},$$

$$LK = LS = \frac{-x + \sqrt{aa+xx}}{\sqrt{2}},$$

$$QN = QT = \frac{-z + \sqrt{aa+zz}}{\sqrt{2}}.$$ Quare in ultimam æquationem introductis hisce valoribus obtinebis

$$a.\overline{IP} - \overline{FI} = \frac{zz}{2} - xx + \frac{bb}{2}$$

$$-\tfrac{1}{4}.\overline{\sqrt{aa+zz} - z}^{\,2} + \tfrac{1}{2}.\overline{\sqrt{aa+xx} - x}^{\,2} - \tfrac{1}{4}.\overline{\sqrt{aa+bb} - b}^{\,2},$$

quæ

quæ redacta ad simpliciorem formam in sequentem mutatur

$$a.\overline{IP-FI} = \frac{z\sqrt{aa+zz}}{2} - x\sqrt{aa+xx} + \frac{b\sqrt{aa+bb}}{2}.$$

Breviter hic mihi liceat adnotare, tangentem parabolæ ad punctum F terminatam a recta CM esse $= \frac{b\sqrt{aa+bb}}{2a}$; similiter tangentes ductas a punctis I, P esse $\frac{x\sqrt{aa+xx}}{2a}$, $\frac{z\sqrt{aa+zz}}{2a}$. Igitur differentia arcuum IP, FI æquabit summam extremarum tangentium, quæ ducuntur a punctis F, P, detracta dupla tangente media, quæ ducitur a puncto I.

Oportet jam determinare z per x. Quoniam est CG : CL :: CL : CQ, habebimus

$$b+\sqrt{aa+bb} : x+\sqrt{aa+xx} :: x+\sqrt{aa+xx} : z+\sqrt{aa+zz} :$$

Ergo $\frac{\overline{x+\sqrt{aa+xx}}^2}{b+\sqrt{aa+bb}} = z+\sqrt{aa+zz}$; sive transposita z, sumptisque quadratis

$$\frac{\overline{x+\sqrt{aa+xx}}^4}{\overline{b+\sqrt{aa+bb}}^2} - 2z.\frac{\overline{x+\sqrt{aa+xx}}^2}{b+\sqrt{aa+bb}} + zz = aa+zz,$$

peractisque opportunis analyseos operationibus

$$\frac{\overline{x+\sqrt{aa+xx}}^4 - a^2.\overline{b+\sqrt{aa+bb}}^2}{\overline{x+\sqrt{aa+xx}}^2} = 2z.\overline{b+\sqrt{aa+bb}}.$$

Quæ æquatio exprimi etiam potest in hunc modum

$$\frac{\overline{x+\sqrt{aa+xx}}^2 + a.\overline{b+\sqrt{aa+bb}}}{x+\sqrt{aa+xx}} . \frac{\overline{x+\sqrt{aa+xx}}^2 - a.\overline{b+\sqrt{aa+bb}}}{x+\sqrt{aa+xx}}$$

$= 2z.\overline{b+\sqrt{aa+bb}}$. In hac æquatione per x remanet z determinata.

Hinc

Hinc colligas velim hujusmodi theorema. Ducta ad principem verticem C parabolæ CIP tangente CM si accipiantur pro libito abscissæ CB $= b$, CH $= x$, tum tertia abscissa CM $= z$, quæ z data est per x, ut supra definitum est, duorum arcuum IP, IF differentia erit rectificabilis, & æqualis

$$\frac{z\sqrt{aa+zz}}{2a} - \frac{x\sqrt{aa+xx}}{a} + \frac{b\sqrt{aa+bb}}{2b}.$$

Ex puncto A ducatur AD normalis assymptoto hyperbolæ. Hactenus accepimus CG > CD; sed si accipiatur C2G < CD; tum recta C2B fieret negativa, adeoque $= -b$. Locum itaque habebit idem theorema, dummodo species b ex positiva convertatur in negativam, & tangens parabolæ ad punctum 2F spectetur ut negativa: quare non erit in æquatione addenda, sed subducenda.

Quod si CG sumatur æqualis CD, fiet $b = o$. Casus hic, cujus solumodo mentionem fecit Comes de Fagnanis, dignus est, qui penitius consideretur. Si supponamus, tres lineas CD, CL, CQ esse continue proportionales, constat, esse

$$a.\overline{IP - CI} = \frac{z\sqrt{aa+zz}}{2} - x\sqrt{aa+xx}.$$ Invenietur autem

$$z = \frac{1}{2a} \cdot \frac{\overline{x+\sqrt{aa+xx}}^2 + aa}{x+\sqrt{aa+xx}} \cdot \frac{\overline{x+\sqrt{aa+xx}}^2 - aa}{x+\sqrt{aa+xx}},$$ & quantitatibus ad secundam potestatem elatis

$$z = \frac{1}{2a} \cdot \frac{2aa+2xx+2x\sqrt{aa+xx}}{x+\sqrt{aa+xx}} \cdot \frac{2xx+2x\sqrt{aa+xx}}{x+\sqrt{aa+xx}},$$

factaque divisione (uterque enim factor divisionem recipit) fiet

$$z = \frac{1}{2a} \cdot 2\sqrt{aa+xx} \cdot 2x = \frac{2x\sqrt{aa+xx}}{a},$$ quæ formula maxima est simplicitate donata.

Progrediens invenio

$$aa + zz = \frac{a^4 + 4a^2x^2 + 4x^4}{a^2}, \ \& \ \sqrt{aa+zz} = \frac{aa+2xx}{a}.$$

Quare $z\sqrt{aa+zz} = \frac{aa+2xx}{aa}.2x\sqrt{aa+xx}$: Ergo

$$a.\overline{IP - CI} = \frac{a^2+2x^2}{aa}.x\sqrt{aa+xx} - x\sqrt{aa+xx} =$$

$$= \frac{2x^3\sqrt{aa+xx}}{aa}.$$

Itaque ſumpta ex arbitratu abſciſſa $CH = x$, accipiatur $CM = \frac{2x\sqrt{aa+xx}}{a}$, invenietur differentia arcuum $IP - CI = \frac{2x^3\sqrt{aa+xx}}{a^3}$. Si $x = a$, fac advertas, fore $CM = 2a\sqrt{2}$, & differentiam arcuum $IP - CI = 2a\sqrt{2}$, atque adeo æqualem CM.

Ex hoc theoremate alia infinita deduci poſſunt ope congruæ combinationis. Aliquot breviter indicabo, quæ methodum patefacient; reliqua induſtriæ relinquam tuæ. Accipe $C\overset{I}{Q}$ quartam proportionalem poſt CG, CL, CQ, & age $\overset{I}{Q}\overset{I}{N}$ normalem aſſymptoto, & ordina rectam $\overset{I}{N}\overset{I}{M}\overset{I}{P}$. Voca $C\overset{I}{M} = \overset{I}{z}$. Theorema duas tibi æquationes ſufficiet.

$$P\overset{I}{P} - IP = \frac{\overset{I}{z}\sqrt{aa+\overset{I}{z}\overset{I}{z}}}{2} - z\sqrt{aa+zz} + \frac{x\sqrt{aa+xx}}{2}$$

$$IP - FI = \frac{z\sqrt{aa+zz}}{2} - x\sqrt{aa+xx} + \frac{b\sqrt{aa+bb}}{2}.$$

Additis duabus æquationibus fiet

$$P\overset{I}{P} - FI = \frac{\overset{I}{z}\sqrt{aa+\overset{I\,I}{zz}}}{2a} - \frac{z\sqrt{aa+zz}}{2a} - \frac{x\sqrt{aa+xx}}{2a} + \frac{b\sqrt{aa+bb}}{2a}.$$

Detracta secunda æquatione ex prima orietur

$P\overset{I}{R} - 2IP + FI$ quantitas algebraica. Hoc modo novas proportionales in assymptoto hyperbolæ capienti combinatio sufficiet quamplurimos arcus parabolicos, quorum differentia rectificabilis est.

Similiter rectificabiles sint differentiæ arcuum $\frac{ip - fi}{IP - FI}$: ergo deducta secunda formula ex prima, invenietur rectificabilis $Pp - Ii - Ii + Ff = Pp - 2Ii + Ff$. Quam ob rem vides, Vir Clarissime, infinitum resultare numerum arcuum, qui si addantur, vel detrahantur, proveniunt algebraice rectificabiles.

Tametsi CG, CL, CQ, $C\overset{I}{Q}$ non sint continue proportionales, dummodo sit $CG : CL :: CQ : C\overset{I}{Q}$, spatia GEKL, $QN\overset{I}{N}\overset{I}{Q}$ erunt æqualia. Eadem propterea methodo algebraica invenietur differentia spatiorum $MN\overset{I}{N}\overset{I}{M}$, BEKH, atque adeo differentia arcuum $P\overset{I}{P}$, FI. Sed hæc sufficiant de parabola, in qua ingens multitudo arcuum, quorum differentia rectificabilis est, non finit, ut omnes distincte recensere valeamus.

Ad ellypsim (*Fig.* 2.) transeo. Sit quadrans ellypticus AKB, cujus semiaxis major CA $= a$, minor CB $= b$. Nosti, vocata CD $= x$, esse arcum BE $= S\,\frac{dx\sqrt{aa - xx.\frac{aa-bb}{aa}}}{\sqrt{aa-xx}}$,

& arcum AE $= S\,\frac{-dx\sqrt{aa - xx.\frac{aa-bb}{aa}}}{\sqrt{aa-xx}}$. Similiter vo-

cata

cata CF = z, erit arcus BG = $S\dfrac{dz\sqrt{aa-zz.\frac{aa-bb}{aa}}}{\sqrt{aa-zz}}$, & arcus AG = $S\dfrac{-dz\sqrt{aa-zz.\frac{aa-bb}{aa}}}{\sqrt{aa-zz}}$.

Pone $z = \dfrac{a\sqrt{aa-xx}}{\sqrt{aa-xx.\frac{aa-bb}{aa}}}$, ex qua, facto non difficili calculo, invenies $x = \dfrac{a\sqrt{aa-zz}}{\sqrt{aa-zz.\frac{aa-bb}{aa}}}$. Quæ formula demonstrat, reciprocari abscissas x, z, idest si facta CD = x fit CF = z, posita CF = x fore CD = z.

Quadrata formula substitutionis erit $zz = \dfrac{a^4 - a^2x^2}{aa-xx.\frac{aa-bb}{aa}}$, aut $a^2z^2 - x^2z^2.\dfrac{aa-bb}{aa} = a^4 - a^2x^2$, transpositisque terminis $aa.\overline{x^2+z^2-a^2} = \dfrac{a^2-b^2}{aa}.x^2z^2$, sumptisque differentiis $a^2.\overline{xdx+zdz} = \dfrac{a^2-b^2}{a^2}xz.Dxz$. Divide per axz, erit $\dfrac{adx}{z} + \dfrac{adz}{x} = \dfrac{a^2-b^2}{a^3}.Dxz$. In primo membro pone valorem z datum per x, in altero valorem x datum per z, & obtinebis

$$\frac{dx\sqrt{aa-xx.\frac{aa-bb}{aa}}}{\sqrt{aa-xx}}+\frac{dz\sqrt{aa-zz.\frac{aa-bb}{aa}}}{\sqrt{aa-zz}}=\frac{a^2-b^2}{a^3}D.xz,$$

factaque integratione

$$S\frac{dx\sqrt{aa-xx.\frac{aa-bb}{aa}}}{\sqrt{aa-xx}}-S\frac{-dz\sqrt{aa-zz.\frac{aa-bb}{aa}}}{\sqrt{aa-zz}}=\frac{a^2-b^2}{a^3}.xz,$$

idest $BE-AG=\frac{\overline{a^2-b^2}.xz}{a^3}=\frac{\sqrt{aa-bb}.\sqrt{zz+xx-aa}}{a}$,

quia supra quantitates istæ æquales inventæ sunt. Integrale hoc modo sumptum completum est, quia facta $x=o$, membra omnia evanescunt.

Hinc sequens theorema habeto. In quocumque quadrante ellyptico AKB abscissa $CD=x$, accipe

$CF=z=\frac{a\sqrt{aa-xx}}{\sqrt{aa-xx.\frac{aa-bb}{aa}}}$; ajo differentiam duorum arcuum BE, AG esse integrabilem, & æqualem

$$\frac{\overline{a^2-b^2}.xz}{a^3}=\frac{\sqrt{a^2-b^2}.\sqrt{z^2+x^2-a^2}}{a}.$$

Si $a=b$, ut ellypsis in circulum convertatur, manifestum est,

$\frac{\overline{a^2-b^2}.xz}{a^3}=o$: Ergo differentia arcuum BE, AG semper nulla erit. Quæ veritas colligitur ex maxime obviis proprietatibus circuli. Nam quum in facta hypothesi sit $CF=z=\sqrt{aa-xx}=DE$, arcus BE, AG æquales sint, necesse est.

Ve-

Verum in ellypſi differentia arcuum B E, A G nulleſcit nunquam exceptis caſibus, in quibus ſit aut $x=o$, aut $x=a$: in quorum primo uterque arcus nullus eſt, in altero uterque coincidit cum quadrante ellyptico.

Ut puncta duo E, G in unum coeant, oportet, eſſe

$$x=z=\frac{a\sqrt{aa-xx}}{\sqrt{aa-xx.\frac{aa-bb}{aa}}}$$, ex qua oritur æquatio

$$a^2x^2-x^4.\frac{a^2-b^2}{a^2}=a^4-a^2x^2$$, quæ reſoluta ſufficit ſequentes radices $x=\frac{a\sqrt{a}}{\sqrt{a-b}}$, $x=\frac{a\sqrt{a}}{\sqrt{a+b}}$. Prima non convenit ellypſi, quia poſtulat $x>a$. Itaque ſeca $CH=\frac{a\sqrt{a}}{\sqrt{a+b}}$, & age perpendicularem HK. Differentia duorum arcuum BK, AK erit quantitas algebraica. Ut ejus valorem obtineas, ſubſtitue in formulis valorem inventum x, ideſt $\frac{a\sqrt{a}}{\sqrt{a+b}}$, & invenies differentiam arcuum $=a-b$, ideſt differentiæ ſemiaxium.

Inter arcus omnes, qui in noſtro canone continentur, duo BK, AK, qui nuper determinati ſunt, gaudent differentia maxima. Etenim caſus maximæ differentiæ poſcit, ut non minus $\frac{dx}{z}+\frac{dz}{x}=o$, quam $z\,dx+x\,dz=o$: Ergo eliminata dz fiet $-\frac{x^2dx}{z}+z\,dx=o$, aut $z^2=x^2$, aut $z=x$. Atqui æqualitas hæc tum ſolum locum habet, qnum $x=CH=\frac{a\sqrt{a}}{\sqrt{a+b}}$: igitur arcus BK, AK præditi ſunt differentia maxima.

Quoniam $BK-AK=a-b$

Item $BE-AG=\frac{aa-bb}{a^3}.xz$, detracta hac ex æquaquatione prima, fiet

$$KE - KG = a - b - \frac{xz.\overline{aa-bb}}{a^3} = \overline{a-b}.\frac{a^3 - xz.\overline{a+b}}{a^3}.$$

Quum reciprocentur abscissæ z, $\overline{x}$, vocata $CF = x$ erit $CD = z$: igitur $BG - AE = \frac{xz.\overline{aa-bb}}{a^3}$, deductaque GE parte comuni, remanebit $BE - AG = \frac{xz.\overline{aa-bb}}{a^3}$, ut antea inventum est.

Accipiatur $Cd = \overset{1}{x}$, $Cf = \overset{1}{z} = \frac{a\sqrt{aa - \overset{1}{x}\overset{1}{x}}}{\sqrt{aa - \overset{1}{x}\overset{1}{x}.\frac{aa-bb}{aa}}}$: erit $Be - Ag = \frac{\overset{1}{x}\overset{1}{z}.\overline{aa-bb}}{a^3}$, præterea $BE - AG = \frac{xz.\overline{aa-bb}}{a^3}$: igitur facta subtractione $Ee - Gg = \frac{\overline{a^2 - b^2}.\overline{\overset{1}{x}\overset{1}{z} - xz}}{a^3}$. Atque hæc sufficiant de arcubus ellypticis, quorum differentia algebraica est.

In hyperbola, de qua mihi unice agendum restat, calculo utemur faciliore. Sit (*Fig.* 3.) hyperbola, cujus semiaxis primus $CA = a$, secundus $= b$, abscissa $CD = x$; cuique cognitum est arcum $AE = S\frac{dx\sqrt{\frac{aa+bb}{aa}xx - aa}}{\sqrt{xx-aa}}$. Similiter si

AF

$AF = z$, arcus

$$AG = S\frac{dz\sqrt{\frac{aa+bb}{aa}zz - aa}}{\sqrt{zz-aa}}.$$ Suppono esse

$$z = \frac{a\sqrt{xx - \frac{a^4}{aa+bb}}}{\sqrt{xx-aa}},$$ ex qua colligo $x = \frac{a\sqrt{zz - \frac{a^4}{aa+bb}}}{\sqrt{zz-aa}}$;

qua de re abscissæ x, z reciprocantur.

Differentiam accipio formulæ xz, & invenio $zdx + xdz = Dxz$. In primo membro pro z substituo ejus valorem datum per x, in secundo valorem x datum per z, & obtineo

$$\frac{adx\sqrt{xx - \frac{a^4}{aa^2+bb}}}{\sqrt{xx-aa}} + \frac{adz\sqrt{zz - \frac{a^4}{a^2+b^2}}}{\sqrt{zz-aa}} = Dxz.$$

Multiplicetur æquatio per $\frac{\sqrt{aa+bb}}{aa}$, & orietur

$$\frac{dx\sqrt{\frac{aa+bb}{aa}xx - aa}}{\sqrt{xx-aa}} + \frac{dz\sqrt{\frac{aa+bb}{aa}zz-aa}}{\sqrt{zz-aa}} = \frac{\sqrt{aa+bb}\,.\,Dxz}{aa},$$

Ergo facto ad finita transitu $AG + AE + M = \frac{xz\sqrt{aa+bb}}{aa}$.

Quantitas M est constans addita in integratione, quæ erit determinanda.

Ad determinandam M parum prodest, supponere $x = a$, quia in hac hypothesi z evadit infinita simul cum arcu AG. Hunc in finem meliorem methodum sequemur, si determinemus

casum, ubi $z = x$. Erit itaque $x = \frac{a\sqrt{xx - \frac{a^4}{aa+bb}}}{\sqrt{xx-aa}}$:

ergo

ergo $x^4 - a^2x^2 = a^2x^2 - \frac{a^6}{a^2+bb}$, qua resoluta exhibet sequentes radices $x = a\sqrt{1+\frac{b}{\sqrt{aa+bb}}}$, $x = a\sqrt{1-\frac{b}{\sqrt{aa+bb}}}$. Secunda in hyperbola locum non habet, poscit enim $x < a$. Itaque seca $CH = a\sqrt{1+\frac{b}{\sqrt{aa+bb}}}$. His positis uterque arcus AE, AG evadit arcus AK. Valoribus istis in inventa æquatione substitutis orietur

$$2AK + M = \frac{\sqrt{aa+\frac{baa}{\sqrt{aa+bb}}}\cdot\sqrt{aa-bb}}{aa} = \sqrt{aa+bb}+b,$$ per quam valorem M obtinemus. Posito autem hoc in æquatione nascetur

$$\begin{matrix} AG - 2AK + AE \\ \text{sive} \\ KG - KE \end{matrix} = \frac{xz\sqrt{aa+bb}}{aa} - \sqrt{aa+bb} - b.$$

Hinc efformatur theorema. Abscissa $CH = a\sqrt{1+\frac{b}{\sqrt{aa+bb}}}$, accipiatur $CD = x$, & $CF = z = \frac{a\sqrt{xx-\frac{a^4}{aa+bb}}}{\sqrt{xx-aa}}$, differentia duorum arcuum hyperbolicorum KG, KE erit algebraica, & æqualis $\frac{xz\sqrt{aa+bb}}{aa} - \sqrt{aa+bb} - b$.

Sume aliam abscissam

$$Cd = \overset{I}{x}, \;\&\; Cf = \overset{I}{z} = \frac{a\sqrt{\overset{I}{x}\overset{I}{x} - \frac{a^4}{a^2+b^2}}}{\sqrt{\overset{I}{x}\overset{I}{x} - aa}},$$

habe-

habebimus $Kg - Ke = \frac{\overset{\text{I}}{x}\overset{\text{I}}{z}.\sqrt{aa+bb}}{aa} - \sqrt{aa+bb} - b$

item $KG - KE = \frac{xz\sqrt{aa+bb}}{aa} - \sqrt{aa+bb} - b$: Ergo detracta secunda æquatione ex prima fiet

$$Gg - Ee = \frac{\overline{\overset{\text{I}}{x}\overset{\text{I}}{z} - xz}.\sqrt{aa+bb}}{aa}.$$

Sine, Vir Doctissime, ut antequam epistolæ finem facio, speciem quamdam paradoxi tibi proponam determinans, ac demonstrans, quænam sit vera differentia inter curvam hyperbolicam, ejusque assymptotum, si producantur in infinitum. Pone $x = a = CA$, ut puncta E, D cadant in puncto A. In hac hypothesi puncta G, F abibunt in infinitum. Quare quum deinceps hæc puncta nominabo, intelligam, ea esse in infinitum remota. Ex nostro theoremate

$KG - KA = \frac{CF.\sqrt{aa+bb}}{a} - \sqrt{aa+bb} - b$: Ergo

$KG - \frac{CF.\sqrt{aa+bb}}{a} = KA - \sqrt{aa+bb} - b$. Addito utrique æquationis parti arcu KA, invenies esse

$$AG - \frac{CF.\sqrt{aa+bb}}{a} = 2KA - \sqrt{aa+bb} - b.$$

Axi CA excita normalem $AL = b$, & duc ALN: constat, hanc esse assymptotum hyperbolæ. Produc FG, donec secet assymptotum in N. Habebis

$AC : CL$, sive $a : \sqrt{aa+bb} :: CF : CN$: ergo

$CN = \frac{CF.\sqrt{aa+bb}}{a}$. Qui valor introductus in nostram formulam suppeditabit

$$AG - CN = 2KA - \sqrt{aa+bb} - b = 2KA - CL - AL.$$

Quæ æquatio ostendit, differentiam inter curvam hyperbolicam in infinitum productam, & ejus assymptotum initium habens in centro, esse æqualem differentiæ inter duplum arcum KA, & summam rectarum LC, LA: hæc autem veritas non raro maximæ esse potest utilitati.

Ex his litteris disces cautionem, qua pronunciandum est, æquationem aliquam non posse ad integrationem perduci. Quicumque æquationem ad hanc formam deduxisset

$$dy = \frac{dx\sqrt{aa - xx.\frac{aa-bb}{aa}}}{\sqrt{aa-xx}} + \frac{dz\sqrt{aa - zz.\frac{aa-bb}{aa}}}{\sqrt{aa-zz}},$$

supposita $z = \frac{a\sqrt{aa-xx}}{\sqrt{aa - xx.\frac{aa-bb}{aa}}}$, nonne affirmaret, ad sui constructionem curvam poscere rectificationem ellypsis? Attamen ipsa est curva algebraica, ut patefaciunt, quæ in hac epistola continentur. De his expecto judicium tuum, quod si, ut spero, aversum non fuerit, pro certo habebo, me veritatem esse assequutum. Vale

Ex Col. S. Luciæ III. Non. Octob. 1755.

DE

DE INTEGRATIONE FORMULÆ

$$\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$$

Per arcus ellypticos, & hyperbolicos. Disquisitio analytica. (a)

QUemadmodum jure optimo utile visum est analystis, eas formulas, quæ algebraicam integrationem non admittunt, ad rectificationem arcus circularis, vel ad logarithmi inventionem reducere, quæ quantitates post algebraicas sunt omnium simplicissimæ: ita pari jure utile esse arbitror, formulas, quæ per quadraturam circuli, aut hyperbolæ minime integrantur, easdem ad rectificationem ellypseos, aut hyperbolæ revocare. Omnium primus cepit hac de re cogitare Comes Julius Carolus de Fagnanis acutissimus analysta, qui arcum lemniscatæ exhibuit per arcum ellypticum, & hyperbolicum, atque hoc pacto elegantiorem reddidit constructionem curvæ isochronæ paracentricæ, quæ a summis viris Jacobo, & Joanne Bernoulliis per rectificationem lemniscatæ absolvitur. Lege ejus opusculum, quod editum est primum in tomo vigesimo nono diarii italici, deinde in tomo secundo ejus operum pag. 343.

Post Fagnanum theoriam hanc vel maxime amplificavit Mac-Laurinus geometra maximus, qui, præter rectificationem lemniscatæ a Fagnano antea traditam, demonstravit, tum binomia $\frac{dz\sqrt{z}}{\sqrt{1\mp zz}}$, $\frac{dz}{\sqrt{z}\cdot\sqrt{1\mp zz}}$, $\frac{dz}{1\mp zz}{}^{\frac{1}{4}}$, $\frac{dz}{1\mp zz}{}^{\frac{3}{4}}$; tum trinomia

(a) *Prodiit hæc disquisitio in collectione Lucensi italico sermone conscripta.*

$\frac{dz\sqrt{z}}{\sqrt{zz+2ez-bb}}$, $\frac{dz\sqrt{z}}{\sqrt{bb+2ez-zz}}$, $\frac{dz\sqrt{z}}{\sqrt{2ez-zz-bb}}$, $\frac{dz}{\sqrt{z}.\sqrt{bb\mp 2ez-zz}}$ supposita rectificatione ellypseos, & hyperbolæ obtineri: quæ inventa exposuit in fluxionum tractatu libro secundo capite tertio.

Mac-Laurinum sequutus est Allembertus vir cum paucis comparandus, qui ostendit, trinomia $\frac{dz\sqrt{z}}{\sqrt{a+bz+czz}}$, $\frac{dz}{\sqrt{z}.\sqrt{a+bz+czz}}$, quæcunque sint quantitates a, b, c vel positivæ, vel negativæ, quoties imaginaria non sunt, semper integrari posita rectificatione arcus ellyptici, & hyperbolici. Hoc demonstratum invenies in Acad. Berolinensi anni 1746. Deinde utilem inventionem adaugens, incredibile dictu est, quantam formularum copiam in eadem Acad. tum anni 1746, tum 1748 ad ea binomia perduxerit, adeo ut theoriam hanc vir ingeniosus propemodum perfecisse dicendus sit.

Quum hæc inventa diligenter, ut par est, considerarem, cognovi, neminem adhuc peculiarem sermonem instituisse de formula $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$, positis f, g, p, q vel positivis, vel negativis, quæ tamen digna est, ut pro virili parte tractetur, tum quia plurimis varietatibus obnoxia est, tum præsertim quia ad eam, quotquot per rectificationem hyperbolæ, & ellypseos integrantur formulæ, revocentur, necesse est. Hoc præstiturus propono mihi integrandam formulam $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$, in qua f, g, p, q sunt quantitates quæcumque vel positivæ, vel negativæ; dummodo nulla ex ipsis $=o$, & in plerisque casibus locum non habeat æqualitas $fq = pg$, in illis scilicet, in quibus posita hac æqualitate radices per divisionem eliduntur. Etenim in hisce suppositionibus formula evaderet algebraice integrategrabilis, aut per quadraturam circuli, vel hyperbolæ construeretur.

In quantitatibus f, g, p, q si signa omnia mutentur, valor formulæ non mutatur, quia formula huic æquivalet

$$\frac{dz\sqrt{-f-gzz}\cdot\sqrt{-1}}{\sqrt{-p-qzz}\cdot\sqrt{-1}}$$, factaque divisione resultat

$$\frac{dz\sqrt{-f-gzz}}{\sqrt{-p-qzz}}$$. Verum si haberem formulam

$$\frac{dz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}$$ mutatis omnibus signis in quantitatibus, quæ subsunt duabus radicibus, formulæ signum $-$ esset præfigendum. Nam ipsa huic æquivalet

$$\frac{dz}{\sqrt{-f-gzz}\cdot\sqrt{-1}\cdot\sqrt{-p-qzz}\cdot\sqrt{-1}}$$, multiplicatisque invicem duabus $\sqrt{-1}$ habebimus

$$\frac{-dz}{\sqrt{-f-gzz}\cdot\sqrt{-p-qzz}}$$. Hæc animadversio probat, formulam non imaginariam esse, sed realem, si utraque radix sit imaginaria, quia mutatis signis omnibus in quantitatibus, quæ signo radicali afficiuntur, utraque radix realis evadit. Verumtamen si una radix realis sit, imaginaria altera, formula sine dubio erit imaginaria. His prænotatis ad rem propius accedo demonstrans, propositam formulam semper integrari rectificata ellypsi, & hyperbola.

Sit quadrans ellypticus A D B, in quo (*Fig* 4.) major C A $= a$, minor C B $= b$. In axe majore accipiatur abscissa C F $= x$, notum est, arcum ellypticum B D $= S\dfrac{dx\sqrt{aa-xx\cdot\frac{aa-bb}{aa}}}{\sqrt{aa-xx}}$.

Adverte formulam hanc integrari quidem ellypsi rectificata, si abscissa x posita sit intra limites $\pm x = o$, $\pm x = a$. Si autem ex his limitibus egrediatur, nondum constat, utrum formula per

per rectificationem ellypsis integretur. Neque putes, eam imaginariam esse, si valor x extra eos limites positus sit. Nam quamquam imaginaria est, si x constituta sit intra limites $\pm x = a$, $\pm x = \frac{aa}{\sqrt{aa-bb}}$; tamen realis est, si statuatur intra novos hos limites $\pm x = \frac{aa}{\sqrt{aa-bb}}$, $\pm x = \infty$. Namque in primo casu radix superior realis est, inferior imaginaria; in altero quum utraque radix imaginaria sit, mutatis omnibus signis, ut antea dictum est, utraque realis evadit Pone $x = cz$, & formula inventa in hanc mutabitur.

$$BD = S\frac{dz\sqrt{aa - cczz.\frac{aa-bb}{aa}}}{\sqrt{\frac{aa}{cc} - zz}}$$. Hanc divide per $\sqrt{e}$, & habebis

$$\frac{BD}{\sqrt{e}} = S\frac{dz\sqrt{aa - cczz.\frac{aa-bb}{aa}}}{\sqrt{\frac{aae}{cc} - ezz}}$$. Quæ formula rectificata ellypsi integratur, si z posita sit intra limites $\pm z = o$, $\pm z = \frac{a}{c}$; imaginaria est, si limites sint $\pm z = \frac{a}{c}$, $\pm z = \frac{aa}{c\sqrt{aa-bb}}$; si vero limites fuerint $\pm z = \frac{aa}{c\sqrt{aa-bb}}$, $\pm z = \infty$, realis erit, sed nondum constat, utrum ad rectificationem ellypsis pertineat.

I. Proponatur jam integranda formula $\frac{dz\sqrt{f-gzz}}{\sqrt{p-qzz}}$. Conferatur cum ea, quæ paullo ante inventa est, & habebitur

tur $a = \sqrt{f}$, $e = q$, $c = \frac{\sqrt{fq}}{\sqrt{p}}$, $b = \frac{\sqrt{fq - gp}}{\sqrt{q}}$: qui valor indicat, non posse formulam referri ad rectificationem ellypsis, nisi fuerit $fq > gp$, quia secus esset imaginarius. Itaque hac conditione habita describatur ellypsis A D B, cujus semiaxis major $AC = \sqrt{f}$, minor $CB = \frac{\sqrt{fq - gp}}{\sqrt{q}}$. In axe majore abscinde $CF = \frac{z\sqrt{fq}}{\sqrt{q}}$, erit $\frac{BD}{\sqrt{q}} = S\frac{dz\sqrt{f - gzz}}{\sqrt{p - qzz}}$. Hæc constructio valet, si z sit intra hos fines $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$.

Si limites fuerint $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, formula erit imaginaria; si vero sint $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \infty$, realis, sed quomodo integretur, nondum compertum est.

Corollarium. Si quæramus, quænam formula construatur per rectificationem ejus ellypsis, in qua semiaxis major est ad minorem ut $\sqrt{2} : 1$, fiet $f = \frac{2fq - 2pg}{q}$, sive $q = \frac{2pg}{f}$, quo valore substituto, factaque opportuna reductione, habebimus $\frac{BD}{\sqrt{2g}} = S\frac{dz\sqrt{f - gzz}}{\sqrt{f - 2gzz}}$.

Iisdem positis accipiatur in axe minore C B abscissa $CG = x$, erit arcus $AD = S\frac{dx\sqrt{bb + xx \cdot \frac{aa - bb}{bb}}}{\sqrt{bb - xx}}$. Hæc autem formula integratur ellypsi rectificata, si x posita sit intra limites $\pm x = o$, $\pm x = b$; imaginaria est semper, si x extra hos limites egrediatur. Fiat, ut supra, $x = cz$, factaque substitutione dividatur æquatio per $\sqrt{e}$, & orietur

$\frac{AD}{\sqrt{e}} = S \frac{dz\sqrt{bb + cczz \cdot \frac{aa - bb}{bb}}}{\sqrt{\frac{bbe}{cc} - ezz}}$: quæ formula exhibetur ab arcu ellyptico, si z media sit intra fines. $\pm z = o$, $\pm z = \frac{b}{c}$; imaginaria est, si $\pm z > \frac{b}{c}$.

II. Oporteat integrare formulam $\frac{dz\sqrt{f + gzz}}{\sqrt{p - qzz}}$. Si comparetur cum ea, quæ nuper inventa est, nascentur hæ determinationes $b = \sqrt{f}$, $e = q$, $c = \frac{\sqrt{fq}}{\sqrt{p}}$, demum $a = \frac{\sqrt{fq + gp}}{\sqrt{q}}$. Quare describe ellypsim, cujus semiaxis major $CA = \frac{\sqrt{fq + gp}}{\sqrt{q}}$, minor $CB = \sqrt{f}$. In hoc sume abscissam $CG = z\frac{\sqrt{fq}}{\sqrt{p}}$, erit $\frac{AD}{\sqrt{q}} = S \frac{dz\sqrt{f + gzz}}{\sqrt{p - qzz}}$. Quapropter hæc formula reducitur ad rectificationem ellypsis, si z contineatur intra fines $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$; erit imaginaria, si $\pm z > \frac{\sqrt{p}}{\sqrt{q}}$.

Corollarium. Si sit $2f = \frac{fq + gp}{q}$, hoc est quadratum axis minoris bis sumptum æquale quadrato axis majoris, sive $q = \frac{gp}{f}$, fiet $\frac{AD}{\sqrt{g}} = S \frac{dz\sqrt{f + gzz}}{\sqrt{f - gzz}}$.

Tran-

Transfeo ad hyperbolam, cujus primus (*Fig.* 5.) femiaxis $KL = a$, fecundus $KM = b$. In primo fumo $KP = x$ Quifque cognofcit, arcum $LN = S \frac{dx \sqrt{-aa + xx \cdot \frac{aa+bb}{aa}}}{\sqrt{-aa+xx}}$. Hanc formulam integratam exhibet arcus hyperbolicus, fi x fit intra limites $\pm x = a$, $\pm x = \infty$. Si x pofita fit intra limites $\pm x = \frac{aa}{\sqrt{aa+bb}}$, $\pm x = a$, fine dubio formula eft imaginaria. At fi limites fuerint $\pm x = o$, $\pm x = \frac{aa}{\sqrt{aa+bb}}$, realis eft, fed nondum conftat, utrum ad rectificationem hyperbolæ reducatur. Fiat $x = cz$, & dividatur æquatio per $\sqrt{e}$, ut oriatur

$$\frac{LN}{\sqrt{e}} = S \frac{dz \sqrt{-aa + cczz \cdot \frac{aa+bb}{aa}}}{\sqrt{-\frac{aae}{cc} + ez^2}}.$$

De qua formula idem dicendum eft ac de fuperiore. Nam fi limites z fint $\pm z = \frac{a}{c}$, $\pm z = \infty$, reducitur ad rectificationem hyperbolæ; fi fint $\pm z = \frac{aa}{c\sqrt{aa+bb}}$, $\pm z = \frac{a}{c}$, imaginaria eft; demum ftatutis limitibus $\pm z = o$, $\pm z = \frac{aa}{c\sqrt{aa+bb}}$ eft realis, fed de ejus integratione per arcus hyperbolicos nondum conftat.

III. Integranda proponatur formula $\frac{dz\sqrt{-f+gzz}}{\sqrt{-p+qzz}}$. Fiat comparatio, & hi valores prodibunt $a = \sqrt{f}$, $e = q$, $c = \frac{\sqrt{fq}}{\sqrt{p}}$, $b = \frac{\sqrt{pg-qf}}{\sqrt{q}}$, qui valor, ne imaginarius fit, poftulat, ut $pg > fq$. Posita hac conditione describe hyperbolam, cujus semiaxis primus $KL = \sqrt{f}$, secundus $KM = \frac{\sqrt{gp - fq}}{\sqrt{q}}$. Sume in primo abscissam $KP = \frac{z\sqrt{fq}}{\sqrt{p}}$, & habebis

$\frac{LN}{\sqrt{q}} = S\frac{dz\sqrt{-f + gzz}}{\sqrt{-p + qzz}}$. Hæc formula ita construitur per arcus hyperbolicos, si z contineatur intra fines $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \infty$: sed z constituta intra limites $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$ est imaginaria: demum si limites fuerint $\pm z = o$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, realis est, sed adhuc dubiæ constructionis.

Corollarium. Ad habendum hyperbolam æquilateram oportet, ut $f = \frac{gp - fq}{q}$, sive $q = \frac{gp}{2f}$: quo valore substituto obtinemus $\frac{LN}{\sqrt{g}} = S\frac{dz\sqrt{-f + gzz}}{\sqrt{-2f + gzz}}$.

Iisdem positis abscissa $KQ = x$ sumatur in secundo axe, & invenietur $LN = S\frac{dx\sqrt{bb + xx . \frac{aa + bb}{bb}}}{\sqrt{bb + xx}}$, quæ formula, quicumque sit valor x, semper rectificato arcu hyperbolico construetur. Statue $x = cz$, & divide æquationem per $\sqrt{c}$, ut obtineas

$$\frac{LN}{\sqrt{c}} = S\frac{dz\sqrt{bb + cczz . \frac{aa + bb}{bb}}}{\sqrt{\frac{bbc}{cc} + czz}}.$$

IV.

IV. Integranda sit formula $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$. Facta comparatione cum superiore reperies $b=\sqrt{f}$, $e=q$, $c=\frac{\sqrt{fq}}{\sqrt{p}}$, $a=\frac{\sqrt{gp-fq}}{\sqrt{q}}$; ex quo valore discis, debere $gp > fq$, ne primus axis proveniat imaginarius. Si in formula adsit hæc conditio, describe hyperbolam, cujus semiaxis primus $KL=\frac{\sqrt{gp-fq}}{\sqrt{q}}$, alter $KM=\sqrt{f}$; tum sume in secundo axe abscissam $KQ=\frac{z\sqrt{fq}}{\sqrt{p}}$, erit $\frac{LN}{\sqrt{q}}=S\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$; quæ formula, quicumque sit valor z, hanc constructionem admittit.

Corollarium. In hyperbola æquilatera fiet $f=\frac{gp-fq}{q}$, sive $q=\frac{gp}{2f}$; quo valore substituto habebimus

$$\frac{LN}{\sqrt{g}}=S\frac{dz\sqrt{f+gzz}}{\sqrt{2f+gzz}}.$$

Ut alias formulas, quæ non exhibent elementum arcus sectionum conicarum, similiter integrem, propono mihi formulam $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$, in qua f, g, p, q negative, & positive accipi possunt, & ad eam transformandam utor substitutione $x=\frac{\sqrt{f+gzz}}{\sqrt{p+qzz}}$, ex qua oritur $z=\frac{\sqrt{f-pxx}}{\sqrt{-g+qxx}}$. His positis manifestum est $D.xz=xdz+zdx$. Substituatur in primo membro pro x ejus valor datus per z, & in secundo pro z ejus valor datus per x, & orietur.

Lemma primnm.

$Dxz = (A)\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}} + (B)\frac{dx\sqrt{f-pxx}}{\sqrt{-g+qxx}}$, ex quo lemmate liquet, formulam B integrari per rectificationem sectionum conicarum, quoties integratur formula A.

V. Sint in lemmate omnes f, g, p, q positivæ, quo in casu, ut traditum est N. IV, formula A, si $gp > fq$, integratur per rectificationem hyperbolæ: Ergo etiam formula B, in qua utile erit mutare signa ad imaginaria vitanda. Hujusmodi autem obtinetur constructio. Describatur hyperbola, cujus semiaxis primus $KL = \frac{\sqrt{gp-fq}}{\sqrt{q}}$, secundus $KM = \sqrt{f}$: tum in secundo axe accipiatur $KQ = \frac{z\sqrt{fq}}{\sqrt{p}} = \frac{\sqrt{-f+pxx}}{\sqrt{g-qxx}} \cdot \frac{\sqrt{fq}}{\sqrt{p}}$, erit

$\frac{LN}{\sqrt{q}} = S\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$: Igitur

$S\frac{dx\sqrt{-f+pxx}}{\sqrt{g-qxx}} = xz - \frac{LN}{\sqrt{q}}$, sive

$S\frac{dx\sqrt{-f+pxx}}{\sqrt{g-qxx}} = \frac{x\sqrt{-f+px^2}}{\sqrt{g-qx^2}} - \frac{LN}{\sqrt{q}}$. Itaque, si $gp > fq$, formula rectificata hyperbola construitur, dummodo x sit intra limites $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$, $\pm x = \frac{\sqrt{g}}{\sqrt{q}}$. Verum si fuerit aut intra limites $\pm x = o$, $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$, aut intra hos alios $\pm x = \frac{\sqrt{g}}{\sqrt{q}}$, $\pm x = \infty$, formula erit imaginaria.

Corollarium. Si ponas $q = \frac{gp}{2f}$, ut habetur in hyperbola æqui-

æquilatera, æquatio in hanc mutabitur

$$S\,\frac{dx\sqrt{f-pxx}}{\sqrt{-2f+pxx}} = \frac{x\sqrt{f-pxx}}{\sqrt{-2f+pxx}} - \frac{LN}{\sqrt{p}}.$$

VI. Si f, g, p positivæ sint, q negativa, lemma superius hanc æquationem præbebit

$$Dxz = (A)\,\frac{dz\sqrt{f+gzz}}{\sqrt{p-qzz}} + (B)\,\frac{dx\sqrt{-f+pxx}}{\sqrt{g+qxx}},$$ existente

$$x = \frac{\sqrt{f+gzz}}{\sqrt{p-qzz}},\ \&\ z = \frac{\sqrt{-f+pxx}}{\sqrt{g+qxx}}.$$ Formula A, ut docuimus N. II semper pertinet ad rectificationem ellypsis, si limites z fuerint $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$: Ergo etiam formula B, si limites x sint $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$, $\pm x = \infty$. Utraque autem formula extra eos limites imaginaria est. Formula B hoc modo construetur. Descripta (*Fig. 4.*) ellypsi, cujus semiaxis major

$CA = \frac{\sqrt{fq+gp}}{\sqrt{q}}$; minor $CB = \sqrt{f}$, & sumpta in hoc abscissa

$$CG = \frac{z\sqrt{fq}}{\sqrt{p}} = \frac{\sqrt{-f+pxx}}{\sqrt{g+qxx}}.\frac{\sqrt{fq}}{\sqrt{p}},$$ erit

$$\frac{AD}{\sqrt{q}} = S\,\frac{dz\sqrt{f+gzz}}{\sqrt{p-qzz}}.$$ Igitur

$$S\,\frac{dx\sqrt{-f+pxx}}{\sqrt{g+qxx}} = xz - \frac{AD}{\sqrt{q}},$$ sive

$$S\,\frac{dx\sqrt{-f+pxx}}{\sqrt{g+qxx}} = \frac{x\sqrt{-f+pxx}}{\sqrt{g+qxx}} - \frac{AD}{\sqrt{q}}.$$

Corollarium. Tunc adhibenda est ellypsis, in quo axes sunt ut $\sqrt{2}:1$, quum $q = \frac{gp}{f}$, quo in casu æquatio nostra in hanc mutatur

S

$$S\frac{dx\sqrt{-f+pxx}}{\sqrt{f+pxx}} = \frac{x\sqrt{-f+pxx}}{\sqrt{f+pxx}} - \frac{AD}{\sqrt{p}}.$$

VII. Quum f, p positivæ ſunt, g, q negativæ, aut viceverſa, ſi $fq > pg$, formula differentialis A integratur ellypſi rectificata, ut habetur N. I, dummodo z media ſit inter limites $\pm z = 0$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$: Ergo etiam altera formula B, dummodo limites x ſint $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$, $\pm x = \infty$. In hac hypotheſi æquatio lemmatis formam hanc aſſumit

$$Dxz = (A)\frac{dz\sqrt{f-gzz}}{\sqrt{p-qzz}} + (B)\frac{dx\sqrt{f-pxx}}{\sqrt{g-qxx}} \text{ exiſtente}$$

$x = \frac{\sqrt{f-gzz}}{\sqrt{p-qzz}}$, & $z = \frac{\sqrt{f-pxx}}{\sqrt{g-qxx}}$. Quoniam formula B eamdem formam habet, ac A, videtur conſtrui poſſe ex N. I eodem modo, ac A. Verum ſi adverteris limites indeterminatarum z, x, cognoſces, utramque formulam non poſſe ex methodo N. I integrari. Sed etiamſi x ſit intra limites $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$, $\pm x = \infty$, noſtra æquatio patefacit, formulam B pertinere ad rectificationem ellypſis. Formula autem B, in qua mutabimus ſigna, ut vitemus imaginaria, hoc modo conſtruitur. Deſcripta ellypſi, cujus ſemiaxis major $CA = \sqrt{f}$, minor $CB = \frac{\sqrt{fq-gp}}{\sqrt{q}}$, accipiatur in primo $CF = \frac{z\sqrt{fq}}{\sqrt{p}} = \frac{\sqrt{-f+pxx}}{\sqrt{-g+qxx}} \cdot \frac{\sqrt{fq}}{\sqrt{p}}$, erit

$$\frac{BD}{\sqrt{q}} = S\frac{dz\sqrt{f-gzz}}{\sqrt{p-qzz}} : \text{Ergo}$$

$$S\frac{dx\sqrt{-f+pxx}}{\sqrt{-g+qxx}} = xz - \frac{BD}{\sqrt{q}}, \text{ ſive}$$

S

$$S\frac{dx\sqrt{-f+pxx}}{\sqrt{-g+qxx}}=\frac{x\sqrt{-f+pxx}}{\sqrt{-g+qxx}}-\frac{BD}{\sqrt{q}}.$$

Corollarium. Si ellypsis axes fuerint, ut $\sqrt{2}:1$, quod habetur in hypothesi $fq=2gp$, hæc prodibit æquatio

$$S\frac{dx\sqrt{-f+pxx}}{\sqrt{-f+2pxx}}=\frac{x\sqrt{-f+pxx}}{\sqrt{-f+2pxx}}-\frac{BD}{\sqrt{2p}}.$$

Quod si $gp>fq$, lemma hanc præbebit æquationem

$$D.xz=(A)\frac{dz\sqrt{-f+gzz}}{\sqrt{-p+qzz}}+(B)\frac{dx\sqrt{-f+pxx}}{\sqrt{-g+qxx}},$$ existente

$$x=\frac{\sqrt{-f+gzz}}{\sqrt{-p+qzz}}, \ \&\ z=\frac{\sqrt{-f+pxx}}{\sqrt{-g+qxx}}.$$ Quoniam x est intra limites $\pm x=\infty$, $\pm x=\frac{\sqrt{g}}{\sqrt{q}}$, si z sit intra limites $\pm z=\frac{\sqrt{p}}{\sqrt{q}}$, $\pm z=\infty$, tam formula A, quam formula B hisce limitibus constitutis integrabitur eadem hyperbola rectificata per regulam traditam N. III. Quapropter nulla nova formula ex nostra æquatione integratur, sed ex ea docemur, summam, aut differentiam duorum arcuum hyperbolicorum esse algebraice rectificabilem. Hanc ob rem (*Fig.* 5.) describatur hyperbola, cujus semiaxis primus $KL=\sqrt{f}$, secundus $KM=\frac{\sqrt{gp-fq}}{\sqrt{q}}$: tum sumatur abscissa $KP=\frac{z\sqrt{fq}}{\sqrt{p}}$, & determinetur arcus LN, erit

$$\frac{LN}{\sqrt{q}}=S\frac{dz\sqrt{-f+gzz}}{\sqrt{-p+qzz}}$$: iterum sumatur abscissa

$KS=\frac{x\sqrt{fq}}{\sqrt{g}}$, ductaque ordinata SO determinetur arcus LO, erit $\frac{LO}{\sqrt{q}}=S\frac{dx\sqrt{-f+pxx}}{\sqrt{-g+qxx}}$. Quare integratio nostræ æqua-

tio-

tionis sufficit $xz = \frac{LN + LO}{\sqrt{q}} + M$.

Si ad determinandam constantem additam M ponerem KP = KL, sive $\frac{z\sqrt{fq}}{\sqrt{p}} = \sqrt{f}$, prodiret KS, seu $\frac{x\sqrt{fq}}{\sqrt{g}}$ infinita. Quare ad determinationem faciendam alia methodo utar, scilicet inveniam abscissam KT, quæ non minus sit æqualis $\frac{z\sqrt{fq}}{\sqrt{p}}$, quam $\frac{x\sqrt{fq}}{\sqrt{g}}$. Itaque habemus

$\frac{z}{\sqrt{p}} = \frac{x}{\sqrt{g}} = \frac{\sqrt{-f+gzz}}{\sqrt{g}\cdot\sqrt{-p+qzz}}$, ex qua formula facto calculo inveniemus $z^4 - \frac{2p}{q}zz = \frac{-fp}{gq}$. Hæc æquatio resoluta dabit $z = \sqrt{\frac{p}{q} + \sqrt{\frac{pp}{qq} - \frac{fp}{gq}}}$, & $z\frac{\sqrt{fq}}{\sqrt{p}} = \sqrt{f + f\cdot\frac{\sqrt{gp - fq}}{\sqrt{gp}}}$, cui quantitati abscindatur æqualis KT, & determinetur arcus LV. In hac hypothesi fiet $x = \sqrt{\frac{g}{q} + \sqrt{\frac{gg}{qq} - \frac{fg}{pq}}}$. Igitur æquatio huic hypothesi accommodata evadet

$$\sqrt{\frac{p}{q} + \sqrt{\frac{pp}{qq} - \frac{fp}{gq}}} \cdot \sqrt{\frac{g}{q} + \sqrt{\frac{gg}{qq} - \frac{fg}{pq}}} = \frac{2LV}{\sqrt{q}} + M.$$

Vocetur primus terminus coalescens ex duabus radicibus simul multiplicatis = F, erit $F - \frac{2LV}{\sqrt{q}} = M$. Quo valore substituto in æquatione habebimus

$$xz - F = \frac{LN + LO - 2LV}{\sqrt{q}}, \text{ sive}$$

$xz - F = \frac{VO - VN}{\sqrt{q}}$. Differentia ergo duorum arcuum hyperperbolicorum V O, V N est algebraice rectificabilis, quam proprietatem ostendi in Epistola data Jacobo Mariscotto V. Cl., in qua nonnulla consectaria non exigui momenti deducta videbis.

Aliam item ejusdem formulæ transformationem obtineo, utens hac methodo. Pono $x = \frac{\sqrt{p+qzz}}{\sqrt{f+gzz}}$, ex qua provenit $z = \frac{\sqrt{p-fxx}}{\sqrt{-q+gxx}}$. Quadrata alterutra ex his formulis ad hanc æquationem pervenio $gzzxx = p + qzz - fxx$, cujus differentiam sumo hoc modo $gzx\,D.zx = qzdz - fxdx$. Divido per zx, & habeo $g\,Dzx = \frac{qdz}{x} - \frac{fdx}{z}$. In prima colloco valorem x datum per z, in secunda valorem z datum per x, & invenio

Lemma Secundum.

$g\,Dzx = (A)\frac{qdz\sqrt{f+gzz}}{\sqrt{p+qzz}} - (B)\frac{fdx\sqrt{-q+gxx}}{\sqrt{p-fxx}}$, quod lemma tradit integrationem formulæ B, quoties integratur formula A.

VIII. Si omnes f, g, p, q, sunt positivæ, formula A ex N. IV integratur rectificata hyperbola, dummodo $gp > fq$: Ergo etiam formula B, si x contineatur intra limites $\pm x = \frac{\sqrt{p}}{\sqrt{f}}$, $\pm x = \frac{\sqrt{q}}{\sqrt{g}}$, extra quos formula est imaginaria. Formula B iisdem conditionibus prædita est, ac illa, quæ integrata est N. V. Quare quum ibi eam constructam dederim, novam constructionem, quæ nascitur ex ultima æquatione, lubens omitto.

Si q tantum positiva sit, reliquæ omnes negativæ, aut vice versa, lemma secundum hanc formam accipiet

$g\,Dzx = -(A)\frac{qdz\sqrt{f+gzz}}{\sqrt{p-qzz}} - (B)\frac{fdx\sqrt{q+gxx}}{\sqrt{p-fxx}}$, existente $x = \frac{\sqrt{p - qzz}}{\sqrt{f + gzz}}$, & $z = \frac{\sqrt{p - fxx}}{\sqrt{q + gxx}}$. Formula B est prorsus similis formulæ A, atque si z media sit intra limites $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, x posita erit intra fines $\pm x = \frac{\sqrt{p}}{\sqrt{f}}$, $\pm x = o$. Quare utraque construitur per regulam traditam N. II. Describatur (*Fig.* 4.) ellypsis ADB, cujus semiaxis major $CA = \frac{\sqrt{fq + gp}}{\sqrt{q}}$, minor $CB = \sqrt{f}$. In hoc accipiatur $CG = \frac{z\sqrt{fq}}{\sqrt{p}}$, & habebitur $\frac{AD}{\sqrt{q}} = S\frac{dz\sqrt{f + gzz}}{\sqrt{p - qzz}}$. Deinde describatur alia ellypsis aeb, (*Fig.* 6.) cujus semiaxis major $ca = \frac{\sqrt{fg + gp}}{\sqrt{f}}$, minor $cb = \sqrt{q}$, atque in hoc accipiatur $ch = \frac{x\sqrt{fq}}{\sqrt{p}}$, erit arcus $\frac{ae}{\sqrt{f}} = S\frac{dx\sqrt{q + gxx}}{\sqrt{p - fxx}}$: Igitur orietur æquatio $gzx = M - AD\sqrt{q}, - ae\sqrt{f}$.

Ut determinetur quantitas addenda M, advertendum est, fieri $CG = \sqrt{f} = CB$, si $ch = \frac{x\sqrt{fq}}{\sqrt{p}} = o$: Ergo nullescente arcu ae, arcus AD fit æqualis quadranti ellyptico AB: quare $M = AB\sqrt{q}$: Ergo $gzx = \overline{AB - AD}.\sqrt{q} - ae\sqrt{f}$, sive $gzx = BD\sqrt{q} - ae\sqrt{f}$.

Duæ ellypses descriptæ similes sunt, quia earum axes eamdem habent proportionem. Quare facile est, utrumque arcum in eadem ellypsi accipere. Secetur CH ita ut sit cb:CB::ch:CH, sive $\sqrt{q} : \sqrt{f} :: \frac{x\sqrt{fq}}{\sqrt{p}} : CH = \frac{fx}{\sqrt{p}}$. Notum est fore cb:CB, seu $\sqrt{q} : \sqrt{f} ::$ ae : AE : Ergo $AE\sqrt{q} = ae\sqrt{f}$: Igitur

gzx

$gzx = BD\sqrt{q} - AE\sqrt{q}$, sive
$\frac{gzx}{\sqrt{q}} = BD - AE$. Itaque differentia duorum arcuum B D, A E rectificabilis est; quam proprietatem demonstravi in epistola ad Mariscottum, sed tum abscissas sumpsi in axe majore. Vides in hoc exemplo quo pacto arcus possis in qualibet ellypsi simili accipere, quod etiam de hyperbola dictum volo. Hæc autem animadversio in plerisque casibus potest esse utilitati.

Si g, q fuerint negativæ, reliquæ positivæ, vel viceversa, lemma hanc æquationem præbebit,

$gDzx = (A)\frac{q\,dz\sqrt{f-gzz}}{\sqrt{p-qzz}} + (B)\frac{f\,dx\sqrt{q-gxx}}{\sqrt{p-fxx}}$. Si ponamus $fq > gp$, ex N. I formula A integratur ellypsi rectificata, si z sit intra limites $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$. Hoc in casu quum sit

$x = \frac{\sqrt{p-qzz}}{\sqrt{f-gzz}}$, & $z = \frac{\sqrt{p-fxx}}{\sqrt{q-gxx}}$, erit x intra limites

$\pm x = \frac{\sqrt{p}}{\sqrt{f}}$, $\pm x = o$. Quare utraque formula ex N. I integratur per rectificationem ellypsis. Describatur ellypsis A D B, cujus semiaxis (*Fig.* 4.) major C A $= \sqrt{f}$, minor C B $= \frac{\sqrt{fq-gp}}{\sqrt{q}}$.

Abscinde in majori C F $= \frac{z\sqrt{fq}}{\sqrt{p}}$, erit $\frac{BD}{\sqrt{q}} = S\frac{dz\sqrt{f-gzz}}{\sqrt{p-qzz}}$.

Describatur item nova ellypsis a e b, cujus (*Fig.* 6.) semiaxis major c a $= \sqrt{q}$, minor c b $= \frac{\sqrt{fq-gp}}{\sqrt{f}}$. Abscinde c i $= \frac{x\sqrt{fq}}{\sqrt{p}}$, habebis $\frac{be}{\sqrt{f}} = S\frac{dx\sqrt{q-gxx}}{\sqrt{p-fxx}}$. Igitur

$gzx = BD\sqrt{q} + be\sqrt{f} - M$. Ut determinetur constans M, adverte fieri arcum be æqualem quadranti ba, si BD nullescat: Ergo $M = ba\sqrt{f}$: Igitur $gzx = BD\sqrt{q} + \overline{be - ba}.\sqrt{f}$, sive $gzx = BD\sqrt{q} - ae\sqrt{f}$.

Ut ambo arcus in eadem ellypsi habeantur, pone $ca : CA :: ci : CI$, sive $\sqrt{q} : \sqrt{f} :: \frac{x\sqrt{fq}}{\sqrt{p}} : CI = \frac{fx}{\sqrt{p}}$, patet fore $\sqrt{q} : \sqrt{f} :: ae : AE$: ergo $ae\sqrt{f} = AE\sqrt{q}$: Igitur $gzx = \overline{BD - AE}.\sqrt{q}$, quæ cum superiore consentit. Consectaria, quæ deduci possunt, lege in litteris ad Mariscottum.

IX. Si supponatur $pg > fq$, ita disponatur æquatio

$$gDzx = (A)\frac{qdz\sqrt{-f+gzz}}{\sqrt{-p+qzz}} + (B)\frac{fdx\sqrt{-q+gxx}}{\sqrt{-p+fxx}}.$$

Quoniam $x = \frac{\sqrt{-p+qzz}}{\sqrt{-f+gzz}}$, & $z = \frac{\sqrt{-p+fxx}}{\sqrt{-q+gxx}}$, si z constituatur intra limites $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \infty$, posita erit x intra limites $\pm x = o$, $\pm x = \frac{\sqrt{q}}{\sqrt{g}}$. Quapropter licet formulæ A, B habeant eamdem formam; tamen si A integratur ex methodo N. III, B ex eadem methodo integrari non potest. Verumtamen etiamsi limites x sint $\pm x = o$, $\pm x = \frac{\sqrt{q}}{\sqrt{g}}$, æquatio nostra demonstrat, formulam B, in qua ad vitanda imaginaria signa mutabimus, integrari rectificata hyperbola per hujusmodi constructionem. Descripta (*Fig.* 5.) hyperbola LN, cujus semiaxis primus $KL = \sqrt{f}$, secundus $KM = \frac{\sqrt{pg - fq}}{\sqrt{q}}$, secetur $KP = \frac{z\sqrt{fq}}{\sqrt{p}} = \frac{\sqrt{+p-fxx}}{\sqrt{+q-gxx}} . \frac{\sqrt{fq}}{\sqrt{p}}$, erit

LN

$\frac{LN}{\sqrt{q}} = S \frac{dz\sqrt{-f+gzz}}{\sqrt{-p+qzz}}$: Ergo habebimus

$S \frac{dx\sqrt{q-gxx}}{\sqrt{p-fxx}} = \frac{gzx}{f} - \frac{LN\sqrt{q}}{f}$, sive

$S \frac{dx\sqrt{q-gxx}}{\sqrt{p-fxx}} = \frac{gx\sqrt{p-fxx}}{f\sqrt{q-gxx}} - \frac{LN\sqrt{q}}{f}$.

Corollarium. Si hyperbola adhibita in constructione fuerit æquilatera, facta scilicet $pg = 2fq$, obtinebimus

$S \frac{dx\sqrt{+p-2fxx}}{\sqrt{p-fxx}} = \frac{2fx\sqrt{p-fxx}}{\sqrt{p-2fxx}} - \frac{LN\sqrt{p}}{f}$.

Quæ formulæ inventæ sunt hactenus, dependent a rectificatione solius ellypsis, vel solius hyperbolæ. Ut eas detegamus, quæ per utriusque curvæ rectificationem absolvuntur, novum lemma est constituendum. Hanc ob rem formulam generalem ita dispono

$$\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}} = \frac{fdz+gzzdz}{\sqrt{f+gzz}.\sqrt{p+qzz}} = \frac{gdz.\overline{fq+gqzz}}{gq\sqrt{f+gzz}.\sqrt{p+qzz}} =$$

$$\frac{dz.\overline{fq-gp}}{q.\sqrt{f+gzz}.\sqrt{p+qzz}} + \frac{gdz.\overline{p+qzz}}{q.\sqrt{f+gzz}.\sqrt{p+qzz}} =$$

$$\frac{dz.\overline{fq-gq}}{q\sqrt{f+gzz}.\sqrt{p+qzz}} + \frac{gdz\sqrt{p+qzz}}{q\sqrt{f+gzz}}.$$ Quapropter obtinemus

Lemma tertium.

$$(A)\frac{\overline{fq-gp}.dz}{\sqrt{f+gzz}.\sqrt{p+qzz}} = (B)\frac{qdz\sqrt{f+gzz}}{\sqrt{p+qzz}} - (C)\frac{gdz\sqrt{p+qzz}}{\sqrt{f+gzz}}.$$

X. Ut utraque formula B, C sit in potestate, aliam hypothesim non invenio, quam supponere negativas g, q. In hac hypothesi æquatio erit

(A) $\frac{dz}{\sqrt{f-gzz}\cdot\sqrt{p-qzz}} = +$ (B) $\frac{qdz\sqrt{f-gzz}}{fq-gp\cdot\sqrt{p-qzz}} -$ (C) $\frac{gdz\sqrt{p-qzz}}{fq-gp\cdot\sqrt{f-gzz}}$. Supponamus $fq > gp$. Formula B integratur per arcum ellypticum, formula C per hyperbolicum. Nam positis limitibus integratur

$\pm z = 0$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, B ex N. I, C ex N. IX

$\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, utraque imaginaria

$\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \infty$, B ex N. VII, C ex N. III.

Hujusmodi autem oritur constructio. Descripta, ellypsi, (*Fig.* 4.) cujus semiaxis major $CA = \sqrt{f}$, minor $CB = \frac{\sqrt{fq-gp}}{\sqrt{q}}$, absinde $CF = \frac{z\sqrt{fq}}{\sqrt{p}}$, erit $\frac{BD}{\sqrt{q}} = S\,\frac{dz\sqrt{f-gzz}}{\sqrt{p-qzz}}$. Similiter descripta hyperbola LN, cujus semiaxis (*Fig.* 5.) primus $KL = \sqrt{g}$, secundus $KM = \frac{\sqrt{fq-gp}}{\sqrt{p}}$, secetur

$KP = \frac{\sqrt{f-gzz}}{\sqrt{p-qzz}} \cdot \frac{\sqrt{gp}}{\sqrt{f}}$, erit

$S\,\frac{dz\sqrt{p-qzz}}{\sqrt{f-gzz}} = \frac{qz\sqrt{f-gzz}}{g\sqrt{p-qzz}} - \frac{LN\sqrt{p}}{g}$. Quocirca fiet

$$S\,\frac{dz}{\sqrt{f-gzz}\cdot\sqrt{p-qzz}} = \frac{qz\sqrt{f-gzz}}{fq-gp\,\sqrt{p-qzz}} + \frac{BD\sqrt{q}}{fq-gp} + \frac{LN\sqrt{p}}{fq-gp}.$$

Hæc constructio valet, si limites indeterminatæ fuerint

$\pm z$

$\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$. Verum si limites sint $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \infty$, ita constructio erit peragenda. Descripta ellypsi, cujus semiaxis major $CA = \sqrt{f}$, minor $CB = \frac{\sqrt{fq - gp}}{\sqrt{q}}$, accipiatur in primo $CF = \frac{\sqrt{-f + gzz}}{\sqrt{-p + qzz}} \cdot \frac{\sqrt{fq}}{\sqrt{g}}$, erit

$$S \frac{dz\sqrt{-f + gzz}}{\sqrt{-p + qzz}} = \frac{z\sqrt{-f + gzz}}{\sqrt{-p + qzz}} - \frac{BD}{\sqrt{q}}.$$

Deinde descripta hyperbola (*Fig. 7.*) ln, cujus primus semiaxis $kl = \sqrt{p}$, secundus $km = \frac{\sqrt{fq - gp}}{\sqrt{g}}$, abscindatur $Kp = \frac{z\sqrt{gp}}{\sqrt{f}}$, habebitur

$$S \frac{dz\sqrt{-p + qzz}}{\sqrt{-f + gzz}} = \frac{ln}{\sqrt{g}}$$

: igitur mutatis opportune signis

$$S \frac{dz}{\sqrt{-f + gzz} \cdot \sqrt{-p + qzz}} = \frac{qz\sqrt{-f + gzz}}{fq - gp \cdot \sqrt{-p + qz^2}} + \frac{BD\sqrt{q}}{fq - gp} + \frac{ln\sqrt{g}}{fq - gp}.$$

Hyperbolæ duæ, quæ in duobus constructionibus usurpatæ sunt, similes sunt, quia axes habent proportionales. Quare proclive erit in utraque constructione eadem hyperbola uti. Fiat ut $kl : KL :: kp : KP$ sive $\sqrt{p} : \sqrt{g} :: \frac{z\sqrt{gp}}{\sqrt{f}} : KP = \frac{gz}{\sqrt{f}}$, erit $\sqrt{p} : \sqrt{g} :: ln : LN$: Ergo $ln\sqrt{g} = LN\sqrt{p}$. Igitur

$$S \frac{dz}{\sqrt{-f + gzz} \cdot \sqrt{-p + qzz}} = \frac{-qz\sqrt{-f + gzz}}{fq - gp \cdot \sqrt{-p + qzz}} + \frac{BD\sqrt{q}}{fq - gp} + \frac{LN\sqrt{p}}{fq - gp}.$$

Corollarium. Si fuerit $fq = 2gp$, hyperbolæ in utraque constructione adhibitæ sunt æquilateræ, ellypses vero habent axes ut $\sqrt{2} : 1$.

For-

Formulam $\frac{dz}{\sqrt{f-gzz}\cdot\sqrt{p-qzz}}$, quæ integrata est N. X, in aliam convertere studeo ope substitutionis $zz = M - xx$, in qua M est quantitas determinanda in operationis progressu. Facta autem substitutione provenit

$$\frac{-xdx}{\sqrt{M-xx}\cdot\sqrt{f-gM+gxx}\cdot\sqrt{p-qM+qxx}}.$$

Duplici modo obtineri potest, ut una ex duabus radicibus extrahi possit, nempe si ponatur vel $M = \frac{f}{g}$, vel $M = \frac{p}{q}$. In prima suppositione formula hæc oritur

$$\frac{-dx}{\sqrt{f-gxx}\cdot\sqrt{\frac{gp-fq}{g}+qxx}}.$$

Quando ponimus $fq > gp$; si fiat $fq - gp = gm$, quod trahit $fq > gm$, habebimus

$$\frac{-dx}{\sqrt{f-gxx}\cdot\sqrt{-m+qxx}}.$$

Hæc realis erit, si x media sit inter limites $\pm x = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm x = \frac{\sqrt{m}}{\sqrt{q}}$, qui respondent limitibus $\pm z = 0$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$: nam si z constituatur intra limites $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \infty$, indeterminata x evadit imaginaria. In altera suppositione, nempe $M = \frac{p}{q}$ nascitur hæc formula

$$\frac{-dx}{\sqrt{p-qxx}\cdot\sqrt{\frac{fq-gp}{q}+gxx}}.$$

Quoniam $fq > gp$ supponitur, si fiat $fq - gp = mq$, obtinetur $\frac{-dx}{\sqrt{p-qxx}\cdot\sqrt{m+gxx}}$.

Realis est formula, si x consistat intra limites $\pm x = \frac{\sqrt{p}}{\sqrt{q}}$,

$\pm x$

$\pm x = o$, qui respondent limitibus $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$; si enim z ponatur intra fines $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \infty$, fit x imaginaria.

XI. Hisce perspectis accipe constructionem formulæ (*Fig.* 4.) $\frac{-dx}{\sqrt{f - gxx} \cdot \sqrt{-m + qxx}}$, in qua $fq > gm$. Describe ellypsim ADB, cujus semiaxis major $CA = \sqrt{f}$, minor $CB = \frac{\sqrt{gm}}{\sqrt{q}}$, abscinde $CF = \frac{\sqrt{f - gxx} \cdot \sqrt{fq}}{\sqrt{fq - gm}}$, & determina arcum BD. Deinde describe hyperbolam, cujus (*Fig.* 5.) semiaxis primus $KL = \sqrt{g}$, secundus $KM = \frac{g\sqrt{m}}{\sqrt{fq - gm}}$, abscinde

$KP = \frac{x\sqrt{fq - gm} \cdot \sqrt{g}}{\sqrt{-m + qxx} \cdot \sqrt{f}}$, & determina arcum LN. His positis nanciscemur

$$S \frac{-dx}{\sqrt{f - gxx} \cdot \sqrt{-m + qxx}} = \frac{-qx\sqrt{f - gxx}}{gm\sqrt{-m + qxx}} + \frac{BD\sqrt{q}}{gm} + \frac{LN\sqrt{fq - gm}}{gm\sqrt{g}}.$$

Sive mutatis omnibus signis

$$S \frac{dx}{\sqrt{f - gxx} \cdot \sqrt{-m + qxx}} = \frac{+qx\sqrt{f - gxx}}{gm\sqrt{-m + qxx}} - \frac{BD\sqrt{q}}{gm} - \frac{LN\sqrt{fq - gm}}{gm\sqrt{g}}.$$

Corollarium. Si $fq = 2gm$, hyperbola est æquilatera, & ellypsis habet axes ut $\sqrt{2} : 1$.

XII. Alterius formulæ $\frac{-dx}{\sqrt{p - qxx} \cdot \sqrt{m + gxx}}$ constructionem habeto. Acceptis semiaxibus $CA = \frac{\sqrt{mq+gp}}{\sqrt{q}}$, $CB = \sqrt{m}$ describe ellypsim (*Fig.* 4.) ADB, & abscindendo $CF = \frac{\sqrt{p-qx^2}\,.\,\sqrt{mq+gp}}{\sqrt{pq}}$ determina arcum BD. Deinde positis semiaxibus (*Fig.* 5.) $KL = \sqrt{g}$, $KM = \frac{\sqrt{mq}}{\sqrt{p}}$ describe hyperbolam LN, & secta abscissa $KP = \frac{\sqrt{m+gxx}\,.\,\sqrt{gp}}{x\sqrt{gp+qm}}$, determina arcum LN. His peractis habebimus

$$S\frac{-dx}{\sqrt{p-qxx}\,.\,\sqrt{m+gxx}} = \frac{-\sqrt{p-qxx}\,.\,\sqrt{m+gxx}}{mqx} + \frac{BD}{m\sqrt{q}} + \frac{LN\sqrt{p}}{mq},$$

sive mutatis signis in omnibus terminis

$$S\frac{dx}{\sqrt{p-qx^2}\,.\,\sqrt{m+gx^2}} = \frac{+\sqrt{p-qxx}\,.\,\sqrt{m+gxx}}{mqx} - \frac{BD}{m\sqrt{q}} - \frac{LN\sqrt{p}}{mq}.$$

Corollarium. Hyperbola erit æquilatera, & ellypseos axes erunt, ut $\sqrt{2} : 1$, si $gp = mq$.

XIII. Nunc revertor ad Lemma tertium, & supponens in eo negativas g, p nanciscor æquationem

$$(A)\ \frac{fq-gp\,.\,dz}{\sqrt{f-gzz}\,.\,\sqrt{-p+qzz}} = (B)\ \frac{qdz\sqrt{f-gzz}}{\sqrt{-p+qzz}} + (C)\frac{gdz\sqrt{-p+qzz}}{\sqrt{f-gzz}}.$$

Si ponatur $fq > gp$, formula A integratur ex N. XI per arcus ellypticos, & hyperbolicos, formula C ex N. V, & VIII per solam rectificationem hyperbolæ: Ergo per arcus sectionum conicarum integrabitur etiam formula B, quæ rea-

lis

lis erit, si z media sit inter fines $\pm z = \frac{\sqrt{p}}{\sqrt{q}}, \pm z = \frac{\sqrt{f}}{\sqrt{g}}$. Ad constructionem acceptis (*Fig.* 4.) semiaxibus $CA = \sqrt{f}$, $CB = \frac{\sqrt{gp}}{\sqrt{q}}$ describe ellypsim, in cujus axe primo fac abscindas $CF = \frac{\sqrt{f-gzz} \cdot \sqrt{fq}}{\sqrt{fq-gp}}$, & definias arcum BD. Tum acceptis semiaxibus (*Fig.* 5.) $KL = \sqrt{g}$, $KM = \frac{g\sqrt{p}}{\sqrt{fq-gp}}$ describe hyperbolam LN, & sumpta $KP = \frac{z\sqrt{fq-gp} \cdot \sqrt{g}}{\sqrt{-p+qzz} \cdot \sqrt{f}}$ determina arcum LN. His positis est

$$S \frac{dz}{\sqrt{f-gzz} \cdot \sqrt{-p+qzz}} = \frac{+qz\sqrt{f-gzz}}{gp\sqrt{-p+qzz}} - \frac{BD\sqrt{q}}{gp} - \frac{LN\sqrt{fq-gp}}{\sqrt{g}}.$$

Postremo novam describe hyperbolam lo, cujus (*Fig.* 7.) semiaxis primus $kl = \frac{\sqrt{fq-gp}}{\sqrt{g}}$, secundus $km = \sqrt{p}$, accipe in secundo axe abscissam $kr = \frac{\sqrt{-p+qzz}}{\sqrt{f-gzz}} \cdot \frac{\sqrt{gp}}{\sqrt{q}}$, & determina arcum lo. His positis erit $S \frac{dz\sqrt{-p+qzz}}{\sqrt{f-gzz}} = \frac{z\sqrt{-p+qzz}}{\sqrt{f-gzz}} - \frac{lo}{\sqrt{g}}$. Quapropter factis opportunis substitutionibus obtinemus

$$S \frac{dz\sqrt{f-gzz}}{\sqrt{-p+qzz}} = \frac{+z\sqrt{f-gzz} \cdot fq-gp}{gp\sqrt{-p+qzz}} - \frac{gz\sqrt{-p+qzz}}{g\sqrt{f-gzz}} - \frac{BD \cdot fq-gp}{gp\sqrt{q}} - \frac{LN \cdot fq-gp \cdot \sqrt{fq-gp}}{gpq\sqrt{g}} + \frac{lo\sqrt{g}}{q}.$$

Quoniam est $\sqrt{g} : \frac{g\sqrt{p}}{\sqrt{fq-gp}} :: \frac{\sqrt{fq-gp}}{\sqrt{g}} : \sqrt{p}$, duæ hyperbolæ adhibitæ in constructione habent axes proportionales, atque adeo similes sunt: igitur constructio per unicam dumtaxat hyperbolam perfici potest. Ut utrumque arcum sumamus in hyperbola l n o, quod elegantius accidit, fiat KL : kl :: KP : kp, sive $\sqrt{g} : \frac{\sqrt{fq-pg}}{\sqrt{g}}$, vel $g : \sqrt{fq-pg} ::$

$$\frac{z\sqrt{fq-gp}\cdot\sqrt{g}}{\sqrt{-p+qzz}\cdot\sqrt{f}} : kp = \frac{z\cdot fq-gp}{\sqrt{-p+qzz}\cdot\sqrt{gf}}$$; cui abscissæ respondet arcus l n. Manifestum est, esse $g : \sqrt{fq-gp} ::$ LN : ln: Ergo LN . $\sqrt{fq-gp} =$ ln . g, sive

$$\frac{LN.\,fq-gp.\sqrt{fq-gp}}{gpq\sqrt{g}} = \frac{ln.\,fq-gp}{pq\sqrt{g}}.$$ Quare facta substitutione habemus

$$S\frac{dz\sqrt{f-gzz}}{\sqrt{-p+qzz}} = \frac{z\sqrt{f-gzz}.\,fq-gp}{gp\sqrt{-p+qzz}} - \frac{gz\sqrt{-p+qzz}}{q\sqrt{f-gzz}}$$

$$-\frac{BD.\,fq-gp}{gp\sqrt{q}} - \frac{ln.\,fq-gp}{pq\sqrt{g}} + \frac{lo.\sqrt{g}}{q}.$$

Corollarium. Si $fq = 2gp$, ellypsis habet axes ut $\sqrt{2} : 1$, hyperbola æquilatera est. Verum in hac hypothesi hoc notandum est, quod in superiori formula duo arcus ejusdem hyperbolæ l n, l o per eamdem quantitatem multiplicati inveniuntur. Quare pro differentia arcuum l n, l o possumus in formula arcum unicum substituere, nempe n o. Quapropter æquatio ultima in hanc mutabitur

$$S\frac{dz\sqrt{f-gzz}}{\sqrt{-f+2gzz}} = \frac{z.\,3f-4gzz}{2\sqrt{-f+2gzz}.\sqrt{f-gzz}} - \frac{BD}{\sqrt{2g}} - \frac{no\sqrt{f}}{2\sqrt{pg}}.$$

XIV. In Lemmate tertio supponamus dumtaxat q negativam, mutatisque signis in omnibus terminis nanciscemur

$$(A)\ \frac{\overline{fq+gp}\cdot dz}{\sqrt{f+gzz}\cdot\sqrt{p-qzz}} = (B)\ \frac{q\,dz\sqrt{f+gzz}}{\sqrt{p-qzz}} + (C)\ \frac{g\,dz\sqrt{p-qzz}}{\sqrt{f+gzz}}$$

Formula A ex N. XII integratur rectificato arcu ellyptico, & hyperbolico, formula B rectificato arcu ellyptico ex N. II: ergo formula C rectificatis sectionibus conicis est in potestate. Si z media sit inter limites $\pm z = o$; $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, formulæ tres reales sunt, secus imaginariæ. Ita disponamus æquationem

$$(C)\ \frac{dz\sqrt{p-qzz}}{\sqrt{f+gzz}} = (A)\ \frac{\overline{fq+gp}\cdot dz}{g\sqrt{f+gzz}\cdot\sqrt{p-qzz}} - (B)\ \frac{q\,dz\sqrt{f+gzz}}{g\sqrt{p-qzz}}.$$

Constructio hæc enascitur. Descripta (*Fig.* 4.) ellypsi, cujus semiaxes sint $CA = \frac{\sqrt{fq+gp}}{\sqrt{q}}$, $CB = \sqrt{f}$, abscinde $CF = \frac{\sqrt{p-qzz}\cdot\sqrt{fq+gp}}{\sqrt{pq}}$, & determina arcum BD. Advertendum est, ordinatam FD, seu CG fieri $= \frac{z\sqrt{fq}}{\sqrt{p}}$: namque habemus $\frac{fq+gp}{q} : f :: \frac{fq+gp}{q} - \left(\frac{fq+gp}{pq}\cdot\overline{p-qzz}\right) : \frac{fqz^2}{p}$.

Deinde positis (*Fig.* 5.) semiaxibus $KL = \sqrt{g}$, $KM = \frac{\sqrt{fq}}{\sqrt{p}}$ describe hyperbolam, & secando $KP = \frac{\sqrt{f+gzz}\cdot\sqrt{gp}}{z\sqrt{fq+gp}}$ determina arcum LN. Erit

$$S\ \frac{dz}{\sqrt{f+gzz}\cdot\sqrt{p-qzz}} = \frac{+\sqrt{f+gzz}\cdot\sqrt{p-qzz}}{fqz} - \frac{BD}{f\sqrt{q}} - \frac{LN\sqrt{p}}{fq}.$$

In ejusdem ellypsis axe secundo sumenda est abscissa $= \frac{z\sqrt{fq}}{\sqrt{p}}$, quæ eadem erit ac superior CH, cui respondet arcus AD. Erit

S

$$S\frac{dz\sqrt{f+gzz}}{\sqrt{p-qzz}}=\frac{AD}{\sqrt{q}}$$: Igitur

$$S\frac{dz\sqrt{p-qzz}}{\sqrt{f+gzz}}=\frac{+\sqrt{f+gzz}\cdot\sqrt{p-qzz}\cdot\overline{fq+gp}}{fgqz}$$

$$\frac{-BD.\overline{fq+gp}}{fg\sqrt{q}}-\frac{AD\sqrt{q}}{g}-\frac{LN.\sqrt{p}.\overline{fq+gp}}{fgq}.$$

Corollarium. Si $fq=gp$, ellypsis habebit axes ut $\sqrt{2}:1$, & hyperbola erit æquilatera: æquatio autem, ejecta specie q, hæc evadet

$$S\frac{dz\sqrt{f-gzz}}{\sqrt{f+gzz}}=\frac{+2\sqrt{f+gzz}\cdot\sqrt{f-gzz}}{gz}-\frac{2BD}{\sqrt{g}}$$

$$-\frac{AD}{\sqrt{g}}-\frac{2LN\sqrt{f}}{g}$$. Perspicuum est, $BD+AD$ esse æqualem quadranti ellyptico, quam quantitatem in præsens omittere possumus, quia in integrali ea quantitas addenda erit, quam circumstantiæ requirent. Est itaque

$$S\frac{dz\sqrt{f-gzz}}{\sqrt{f+gzz}}=\frac{2\sqrt{f+gzz}\cdot\sqrt{f-gzz}}{gz}-\frac{BD}{\sqrt{g}}-\frac{2LN\sqrt{f}}{g}.$$

XV. Determinatis his duabus formulis, quæ dependent tum ab ellypseos, tum ab hyperbolæ rectificatione, revoco lemma primum, in quo suppono g, p negativas, ut habeam

$$Dxz=(A)\frac{dz\sqrt{f-gzz}}{\sqrt{-p+qzz}}+(B)\frac{dx\sqrt{f+pxx}}{\sqrt{g+qxx}}$$ existente

$$x=\frac{\sqrt{f-gzz}}{\sqrt{-p+qzz}}, \ \& \ z=\frac{\sqrt{f+pxx}}{\sqrt{g+qxx}}$$. Si $fq>gp$ formula A ex N. XIII integratur per rectificationem utriusque sectionis: Ergo etiam formula B. Si z constituatur inter limites

$$\pm z=\frac{\sqrt{p}}{\sqrt{q}}, \pm z=\frac{\sqrt{f}}{\sqrt{g}}$$, quibus positis formula A potest esse realis, erit x intra fines $\pm x=\infty$, $\pm x=o$; quod indicat

cat formulam B ſemper eſſe realem. Accipe conſtructionem. Deſcribe ellypſim, cujus ſemiaxes (*Fig.* 4.) ſint

$$CA = \sqrt{f},\ CB = \frac{\sqrt{gp}}{\sqrt{q}}, \text{ \& abſciſſa}$$

$$CF = \frac{\sqrt{f-gzz}\cdot\sqrt{fq}}{\sqrt{fq-gp}} = \frac{x\sqrt{fq}}{\sqrt{g+qxx}}$$ determina arcum BD.

Tum deſcribe hyperbolam lo, cujus ſemiaxis (*Fig.* 7.) primus

$$kl = \frac{\sqrt{fq-gp}}{\sqrt{g}},$$ ſecundus $km = \sqrt{p}$. Sume in primo

$$kp = \frac{z\cdot\overline{fq-gp}}{\sqrt{gf}\cdot\sqrt{-p+qzz}} = \frac{\sqrt{f+pxx}\cdot\sqrt{fq-gp}}{\sqrt{gf}},$$ & determina arcum ln; item ſume in axe ſecundo

$$kr = \frac{\sqrt{-p+qzz}\cdot\sqrt{gp}}{\sqrt{f-gzz}\cdot\sqrt{q}} = \frac{\sqrt{gp}}{x\sqrt{q}},$$ & determina arcum lo.

Habebimus

$$S\frac{dz\sqrt{f-gzz}}{\sqrt{-p+qzz}} = +\frac{z\sqrt{f-gzz}\cdot\overline{fq-gp}}{gp\sqrt{-p+qzz}} - \frac{gz\sqrt{-p+qzz}}{q\sqrt{f-gzz}}$$
$$-\frac{BD\cdot\overline{fq-gp}}{gp\sqrt{q}} - \frac{ln\cdot\overline{fq-gp}}{pq\sqrt{g}} + \frac{lo\sqrt{g}}{q}.$$ Ergo

$$S\frac{dx\sqrt{f+pxx}}{\sqrt{g+qxx}} = xz - \frac{z\sqrt{f-gzz}\cdot\overline{fq-gp}}{gp\sqrt{-p+qzz}} + \frac{gz\sqrt{-p+qzz}}{q\sqrt{f-gzz}}$$
$$+\frac{BD\cdot\overline{fq-gp}}{gp\sqrt{q}} + \frac{ln\cdot\overline{fq-gp}}{pq\sqrt{g}} - \frac{lo\sqrt{g}}{q}.$$ Demum pro z ſubſtituto ejus valore dato per x, factaque reductione naſcitur

$$S\frac{dx\sqrt{f+pxx}}{\sqrt{g+qxx}} = \frac{g^2p+2gpqxx-fqqx}{gpqx}\cdot\frac{\sqrt{f+px^2}}{\sqrt{g+qxx}}$$
$$+\frac{BD\cdot\overline{fq-gp}}{gp\sqrt{q}} + \frac{ln\cdot\overline{fq-gp}}{pq\sqrt{g}} - \frac{lo\sqrt{g}}{q}.$$

Co-

Corollarium. Si $fq = 2gp$, ellypsis prædita erit axibus, qui erunt in porportione $\sqrt{2} : 1$, hyperbola autem æquilatera erit. Æquatio vero hanc simpliciorem formam induet

$$S \frac{dx\sqrt{f+pxx}}{\sqrt{f+2pxx}} = \frac{f\sqrt{f+pxx}}{2px\sqrt{f+2pxx}} + \frac{BD}{\sqrt{2p}} - \frac{no\sqrt{f}}{2p}.$$

Si in Lemmate primo ponas g negativam habebis

$$Dxz = (A)\frac{dz\sqrt{f-gzz}}{\sqrt{p+qzz}} + (B)\frac{dx\sqrt{f-pxx}}{\sqrt{g+qxx}} \text{ existente}$$

$$x = \frac{\sqrt{f-gzz}}{\sqrt{p+qzz}},\ z = \frac{\sqrt{f-pxx}}{\sqrt{g+qxx}}.$$ Si z constituatur inter limites $\pm z = 0$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, invenietur x esse intra limites $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$, $\pm x = 0$, extra quos limites utraque formula imaginaria est. Finibus hisce sancitis utraque formula integrationem recipit ex N. XIV. Quocirca cognosces, plures arcus ellypticos, & hyperbolicos conjunctos per signa $+ -$ esse algebraice rectificabiles. Verum re diligenter perpensa comperies, nullos arcus haberi alios, nisi eos, quorum differentiæ rectificabiles sunt, de quibus antea loquuti sumus.

XVI. Advoco nunc Lemma secundum, & supponens negativas f, q, sive quod idem est g, p, invenio

$$gDzx = -(A)\frac{qdz\sqrt{f-gzz}}{\sqrt{-p+qzz}} - (B)\frac{fdx\sqrt{q+gxx}}{\sqrt{p+fxx}}.$$ Si $fq > gp$ formula A per arcus utriusque sectionis integratur ex N. XIII: Ergo etiam formula B, quæ eamdem formam habet, ac illa, quæ integrata est N. XV. Quare ejusdem formulæ ex ultima nostra æquatione novam, si optas, potes elicere constructionem.

XVII. Denique in lemmate secundo supponamus g negativam, sive quod idem est g positivam, reliquas omnes negativas. Orietur æquatio

$$gDxz = -(A)\frac{qdz\sqrt{f-gzz}}{\sqrt{p+qzz}} + (B)\frac{fdx\sqrt{q+gxx}}{\sqrt{-p+fxx}}$$

exi-

existente $x = \frac{\sqrt{p+qzz}}{\sqrt{f-gzz}}$, & $z = \frac{\sqrt{-p+fxx}}{\sqrt{q+gxx}}$. Si z constituatur intra limites $\pm z = o$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, habebit x limites $\pm x = \frac{\sqrt{p}}{\sqrt{f}}$, $\pm x = \infty$. Atqui positis his limitibus formula A ex N. XIV integratur per arcus utriusque sectionis: Ergo etiam formula B, quæ formula imaginaria est, si x ex statutis finibus egrediatur. Constructio autem hæc nascitur. Descripta ellypsi, cujus (*Fig.* 4.) semiaxis $CA = \frac{\sqrt{fq+gp}}{\sqrt{g}}$, $CB = \sqrt{p}$, abscinde $CF = \frac{\sqrt{f-gzz}\cdot\sqrt{fq+gp}}{\sqrt{fg}} = \frac{fq+gp}{\sqrt{fg}\cdot\sqrt{q+gxx}}$, cui respondet ordinata $FD = CG = \frac{z\sqrt{gp}}{\sqrt{f}} = \frac{\sqrt{-p+fxx}}{\sqrt{q+gxx}}\cdot\frac{\sqrt{gp}}{\sqrt{f}}$, & determina arcum BD. Deinde positis (*Fig.* 5.) semiaxibus $KL = \sqrt{q}$, $KM = \frac{\sqrt{pg}}{\sqrt{f}}$ describe hyperbolam, & secans $KP = \frac{\sqrt{p+qzz}\cdot\sqrt{fq}}{z\sqrt{fq+gp}} = \frac{x\sqrt{fq}}{\sqrt{-p+qxx}}$ determina arcum LN. In ejusdem ellypsis axe secundo sumenda est abscissa $\frac{z\sqrt{gp}}{\sqrt{f}} = \frac{\sqrt{-p+fxx}}{\sqrt{q+gxx}}\cdot\frac{\sqrt{gp}}{\sqrt{f}}$, quæ est eadem CG, quæ antea determinata est, cui respondet arcus AD. Constat esse

$$S\frac{dz\sqrt{f-gzz}}{\sqrt{p+qzz}} = +\frac{\sqrt{p+qzz}\cdot\sqrt{f-gzz}\cdot fq+gp}{gpqz} - \frac{BD.fq+gp}{pq\sqrt{g}} - \frac{AD\sqrt{g}}{q} - \frac{LN.\sqrt{f}.fq+gp}{gpq}.$$

Quapropter facta substitutione

$$S\frac{dx\sqrt{q+gxx}}{\sqrt{-p+fxx}}=\frac{gxz}{f}+\frac{\sqrt{p+qzz}\cdot\sqrt{f-gzz}\cdot\overline{fq+gp}}{fgpz}$$

$$-\frac{BD.\overline{fq+gp}}{fp\sqrt{g}}-\frac{AD\sqrt{g}}{f}-\frac{LN.\overline{fq+gp}}{gp\sqrt{f}}$$, five pro z substituto ejus valore dato per x

$$S\frac{dx\sqrt{q+gxx}}{\sqrt{-p+fxx}}=\frac{gx\sqrt{-p+fxx}}{f\sqrt{q+gxx}}+$$

$$\frac{x.\overline{fq+gp}^2}{fpg.\sqrt{q+gxx}.\sqrt{-p+fxx}}$$

$$-\frac{BD.\overline{fq+gp}}{fp\sqrt{g}}-\frac{AD\sqrt{g}}{f}-\frac{LN.\overline{fq+gp}}{pg\sqrt{f}}.$$

Corollarium. Si ponatur $fq=gp$, ellypsis prædita erit axibus, qui servant proportionem $\sqrt{2}:1$, hyperbola autem erit æquilatera. Æquatio autem hanc formam accipiet

$$S\frac{dx\sqrt{p+fxx}}{\sqrt{-p+fxx}}=\frac{x\sqrt{-p+fxx}}{\sqrt{p+fxx}}+\frac{4px}{\sqrt{p+fxx}.\sqrt{-p+fxx}}$$

$$-\frac{2BD}{\sqrt{f}}-\frac{AD}{\sqrt{f}}-\frac{2LN}{\sqrt{f}}.$$

Verum quum BD + AD æquet quadrantem ellypticum, tuto omitti potest, quia opportuna quantitas in integratione addenda est. Quare habebimus

$$S\frac{dx.\sqrt{p+fxx}}{\sqrt{-p+fxx}}=\frac{x\sqrt{-p+fxx}}{\sqrt{p+fxx}}+\frac{4px}{\sqrt{p+fxx}.\sqrt{-p+fxx}}-$$

$$\frac{BD}{\sqrt{f}}-\frac{2LN}{\sqrt{f}}.$$

Post has demonstrationes in aperto est, formulam

$\frac{dz\sqrt{f+gzz}}{\sqrt{p+qxx}}$, quæcumque sint f, g, p, q vel positivæ, vel negativæ, semper integrari ellypsi, & hyperbola rectificatis. Ut autem

autem facilius invenire possis, quo in loco hujusce desquisitionis quilibet casus contineatur, sequentem tabulam formavi, quæ indicis locum tenere potest, in qua omnes casus distincti sunt, & numeri expositi, ubi singuli construuntur. Littera E indicabit formulam a rectificatione ellypsis, littera H a rectificatione hyperbolæ dependere. Ubi utraque littera conjungitur, scias, formulam postulare utriusque sectionis rectificationem.

T A B U L A.

$+f+g+p+q; \quad -f-g-p-q$

Si $gp > fq$ H. IV

Si $fq > gp$ E. H. XV, XVI

- - - - - - - - - - - - - - - - - -

$+f+g+p-q; \quad -f-g-p+q$

Si limites sint $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$. E. II

extra hos limites imaginaria.

- - - - - - - - - - - - - - - - - -

$+f+g-p+q; \quad -f-g+p-q$

Si limites sint $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$ imaginaria

$\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \infty$. E. H. XVII

- - - - - - - - - - - - - - - - - -

$+f-g+p+q; \quad -f+g-p-q$

Si limites sint $\pm z = o$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$. E. H. XIV

$\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \infty$ imaginaria

- - - - - - - - - - - - - - - - - -

$-f+g+p+q; \quad +f-g-p-q$

Si limites sint $\pm z = o$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$ imaginaria

$\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \infty$. E.VI

- -

$+f+g-p-q; \quad -f-g+p+q$

Semper imaginaria

- -

$+f-g+p-q; \quad -f+g-p+q$

Si li- $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$. E.I

Si $fq > gp$ mites $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$ imaginaria

sint $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \infty$. E.VII

Si li- $\pm z = o$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$. H.IX

Si $gp > fq$ mites $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$ imaginaria

sint $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \infty$. H.III

- -

$-f+g+p-q; \quad +f-g-p+q$

Si $gp > fq$ H.V, VIII

& limites sint $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$.

Si $fq > gp$ E.H.XIII.

extra hos limites imaginaria.

Quinam

Quinam sit usus hujusce tabulæ, opus non est ut explicem. Nam proposita aliqua formula inveniendus in tabula casus, ubi species f, g, p, q eodem signo donatæ sunt. Post invenies conditiones, in quibus formula aut imaginaria est, aut pertinet ad rectificationem alterutrius, vel utriusque sectionis conicæ, & numeros denotantes, quo in loco disquisitionis hoc sit demonstratum. Caussa exempli proposita formula $\frac{dz\sqrt{f-gzz}}{\sqrt{-p+qzz}}$. Fac invenias in tabula casum, ubi f, q signo $+$, g, p signo $-$ affectæ sunt: atque hic ultimum in tabula locum tenet. Hic notatum reperies, formulam realem esse non posse, nisi z media sit intra limites $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$. Præterea cognosces, si $gp > fq$, formulam integrari hyperbola rectificata, atque hoc demonstrari N. V aut N. VIII. Contra si $fq > gp$, comperies, formulam ad sui integrationem indigere rectificatione utriusque sectionis, atque hoc probari N. XIII. Ita de casibus reliquis.

Ex his omnibus colligas velim, formulas omnes, quæ reducuntur ad nostram, integrari per hyperbolæ & ellypseos rectificationem. Verum de his nihil dico, quia hoc solum mihi in præsentia demonstrandum proposivi, formulam $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$, quæcumque sint f, g, p, q semper construi rectificatis sectionibus conicis: quod quum absolute perfecerim, disquisitioni finem impono.

DE

DE FORMULIS:

Quarum integratio dependet a rectificatione ellypsis, & hyperbolæ. Disquisitio Analytica.

IN superiore disquisitione, ubi per arcus ellypticos, & hyperbolicos construendam curvavi formulam $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$, quam deinceps canonicam appellabo, dixi, formulas omnes, quæ ad hanc reduci possunt, rectificatis hyperbola, & ellypsi similiter construi. In hac vero quænam sint hujusmodi formulæ, investigabo. De theoria hac ingeniose ante me egit Alembertus formulas reducens non ad meam canonicam, sed ad duas alias formulas, quemadmodum monui in superiore disquisitione. Ne discedam ab illis honestis moribus, qui geometram quemque decent, palam profiteor, me non solum ab inventis Analystæ illius plurimam utilitatem cepisse, sed etiam aliquando ipsius methodis usum fuisse: neque ipse profecto tantum fortasse theoriam amplificassem, nisi mihi vir summus facem ante prætulisset. Nihilo tamen secius supervacanea, ut spero, non erunt studia, laboresque mei: nam sæpe methodo simplicitatem, atque elegantiam conciliavi, & theoriam ipsam promovi addens nonnulla, quæ non videntur contemnenda.

I. His præmissis rem aggrediens adverto, me in lemmate tertio ejus disquisitionis, in qua per arcus ellypticos, & hyperbolicos integravi formulam $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$, quam, ut dixi, appellabo canonicam, ad hanc ipsam deduxisse formulam aliam $\frac{dz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}$ usum hac methodo. Formulam canonicam ita dispono

$$\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}=\frac{fdz+gzzdz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}=\frac{fqdz+gqz^2dz}{q\cdot\sqrt{f+gzz}\cdot\sqrt{p+qzz}}$$

$$=\frac{fq-gp\cdot dz}{q\sqrt{f+gzz}\cdot\sqrt{p+qzz}}+\frac{gdz\sqrt{p+qzz}}{q\sqrt{f+gzz}}$$: Igitur

$$\frac{dz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}=\frac{qdz\sqrt{f+gzz}}{fq-gp\cdot\sqrt{p+qzz}}-\frac{gdz\sqrt{p+qzz}}{fq-gp\cdot\sqrt{f+gzz}}.$$

Quæ duæ formulæ quoniam conveniunt cum canonica, constat, formulam $\frac{dz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}$ semper reduci ad canonicam.

Scholium primum. Si una ex radicibus per constantem multiplicata alteram daret, tum $fq=gp$: quare inutilis est hujusmodi præparatio. Verum in hoc casu evidens est, formulam non indigere arcubus ellypticis, & hyperbolicis, sed integrari suppositis dumtaxat quadraturis circuli, & hyperbolæ.

Scholium alterum. In memorata disquisitione N. X, XI, XII tres formulas summatas exhibui, quæ in præsenti continentur. Quare in illis casibus utilius erit eam integrationem usurpare.

II. Si radices duas formulæ superioris simul multiplicemus, aliam obtinebimus hujus formæ $\frac{dz}{\sqrt{a+bzz+cz^4}}$, in qua potest etiam esse $b=o$. Si trinomium $a+bzz+cz^4$, sit resolubile in duo binomia realia hujus formæ $g+zz$, constat ex N. I, formulam integrari supposita rectificatione ellypsis, & hyperbolæ. Verum si in hujusmodi binomia resolubile non sit, quod contingit, quum $a>\frac{bb}{4c}$, docendum est, quo pacto formula sit tractanda. Primum ejiciendus est ex trinomio secundus terminus ope substitutionis $zz+\frac{b}{2c}=uu$, ex qua nascitur

$zz=$

$$zz = uu - \frac{b}{2c}, \; dz = \frac{udu}{\sqrt{uu - \frac{b}{2c}}},$$

$$\sqrt{a + bzz + cz^4} = \sqrt{a - \frac{bb}{4c} 4cu^4}$$

sive positis $\frac{b}{2c} = n$, $a - \frac{bb}{4c} = cm$,

$$zz = uu - n, \; dz = \frac{udu}{\sqrt{uu - n}}, \sqrt{a + bzz + cz^4} = \sqrt{c . m + u^4}.$$

Transformata itaque formula habebimus

$$\frac{dz}{\sqrt{a + bzz + cz^4}} = \frac{udu}{\sqrt{u^2 - n} . \sqrt{c . m + u^4}}.$$

Usurpanda nunc est secunda substitutio

$uu + \sqrt{m + u^4} = yy$, ex qua oritur

$m + u^4 = y^4 - 2uuyy + u^4$, sive

$uu = \frac{y^4 - m}{2yy} = \frac{y^2}{2} - \frac{m}{2y^2}$, & sumptis differentiis

$udu = \frac{ydy}{2} + \frac{mdy}{2y^3} = dy . \frac{y^4 + m}{2y^3}$. Præterea

$uu - n = \frac{y^2}{2} - \frac{m}{2y^2} - n = \frac{y^4 - 2ny^2 - m}{2y^2}$. Hoc trinomium $y^4 - 2ny^2 - m$ resolubile semper est in duo binomia realia, nempe $y^2 - n + \sqrt{n^2 + m}$, $y^2 - n - \sqrt{n^2 + m}$, sive substitutis valoribus m, n

$y^2 - \frac{b}{2c} + \sqrt{\frac{a}{c}}$, $y^2 - \frac{b}{2c} - \sqrt{\frac{a}{c}}$, quæ duo binomia in

no-

nostra hypothesi sunt semper realia, quia quum debeat esse $\frac{a}{c} > \frac{bb}{4cc}$, necessario $\frac{a}{c}$ erit positiva: Ergo

$$\sqrt{u^2 - n} = \frac{\sqrt{yy - \frac{b}{2c} + \sqrt{\frac{a}{c}}} \cdot \sqrt{yy - \frac{b}{2c} - \sqrt{\frac{a}{c}}}}{y\sqrt{2}}.$$

Demum $u^4 + m = \frac{y^4}{4} - \frac{2m}{4} + \frac{m^2}{4y^4} + m = \frac{y^4}{4} + \frac{2m}{4} + \frac{m^2}{4y^4}$:

Ergo $\sqrt{u^4 + m} = \frac{y^2}{2} + \frac{m}{2y^2} = \frac{y^4 + m}{2y^2}$. Itaque peractis substitutionibus fiet

$$\frac{dz}{\sqrt{a + bzz + cz^4}} = \frac{dy\sqrt{2}}{\sqrt{c} \cdot \sqrt{yy - \frac{b}{2c} + \sqrt{\frac{a}{c}}} \cdot \sqrt{y^2 - \frac{b}{2c} - \sqrt{\frac{a}{c}}}},$$

quæ ex N. I ad canonicam reducitur. Facto autem calculo invenies

$$\frac{dz}{\sqrt{a + bz^2 + cz^4}} = \frac{dy\sqrt{\frac{-b}{2c} + \sqrt{\frac{a}{c}} + yy}}{\sqrt{2a} \cdot \sqrt{\frac{-b}{2c} - \sqrt{\frac{a}{c}} + yy}} - \frac{dy\sqrt{\frac{-b}{2c} - \sqrt{\frac{a}{c}} + yy}}{\sqrt{2a} \cdot \sqrt{\frac{-b}{2c} + \sqrt{\frac{a}{c}} + yy}},$$

existente $y = \sqrt{zz + \frac{b}{2c} + \sqrt{\frac{a + bz^2 + cz^4}{c}}}$.

III. Ut ad canonicam redigam formulam $\frac{zz\,dz}{\sqrt{f + gzz} \cdot \sqrt{p + qzz}}$, deduco formulam numeri I a canonica hoc modo

$$\frac{dz\sqrt{f + gzz}}{\sqrt{p + qzz}} - \frac{f\,dz}{\sqrt{f + gzz} \cdot \sqrt{p + qzz}} = \frac{gzz\,dz}{\sqrt{f + gzz} \cdot \sqrt{p + qzz}}:$$

Igitur

$$\frac{zzdz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}} = \frac{dz\sqrt{f+gzz}}{g\sqrt{p+qzz}} - \frac{fdz}{g\sqrt{f+gzz}\cdot\sqrt{p+qzz}}:$$

atqui ex N. I

$$\frac{fdz}{g\sqrt{f+gzz}\cdot\sqrt{p+qzz}} = \frac{fqdz\sqrt{f+gzz}}{g\cdot fq-gp\cdot\sqrt{p+qzz}} - \frac{fdz\sqrt{p+qzz}}{fq-gp\cdot\sqrt{f+gzz}}$$

: Ergo facta subtractione

$$\frac{zzdz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}} = \frac{-pdz\sqrt{f+gzz}}{fq-gp\cdot\sqrt{p+qzz}} + \frac{fdz\sqrt{p+qzz}}{fq-gp\cdot\sqrt{f+gzz}},$$

quæ duæ coincidunt cum canonica.

Scholium. Inutilis est hæc præparatio, si $fq = gp$; sed formula in hoc casu ad sui integrationem non indiget, nisi circuli, aut hyperbolæ quadratura.

IV. Ut ad canonicam reducam formulam $\frac{zzdz}{\sqrt{a+bzz+cz^4}}$, quum trinomium $a+bzz+cz^4$ non potest resolvi in duo binomia realia hujus formæ $f+gzz$, necesse est, ut prius ad canonicam perducam formulam $\frac{dz\sqrt{f+gzz}\cdot\sqrt{p+qzz}}{zz}$. Hanc ob rem utar methodo, quam deinceps latius patere ostendam. Quantitatis $\frac{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}{z}$ accipio differentiam hoc modo

$$D\frac{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}{z} = \frac{qdz\sqrt{f+gzz}}{\sqrt{p+qzz}} + \frac{gdz\sqrt{p+qzz}}{\sqrt{f+gzz}} - \frac{dz\sqrt{f+gzz}\cdot\sqrt{p+qzz}}{zz}$$

: Ergo facta transpositione

$$\frac{dz\sqrt{f+gzz}\cdot\sqrt{p+qzz}}{zz} = -D\frac{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}{z} + \frac{qdz\sqrt{f+gzz}}{\sqrt{p+qzz}} + \frac{gdz\sqrt{p+qzz}}{\sqrt{f+gzz}}.$$

Quare proposita formu-

la

la inventa est æqualis duabus, quæ conveniunt cum canonica, dempta formula integrabili.

V. Ad reducendam formulam $\frac{zz\,dz}{\sqrt{a+bzz+cz^4}}$, quum $\frac{a}{c} > \frac{bb}{4cc}$, quo in casu trinomium non est resolubile in binomia realia, utor eadem methodo, iisdemque substitutionibus, quibus usus sum N. II. Effecta prima substitutione invenies $\frac{zz\,dz}{\sqrt{a+bzz+cz^4}} = \frac{u\,du\sqrt{uu-n}}{\sqrt{c.\,m+u^4}}$. Peracta secunda oritur

$$\frac{zz\,dz}{\sqrt{a+bzz+cz^4}} = \frac{dy\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}\,.\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{\sqrt{2c}\,.\,yy}$$

atqui ex N. IV.

$$\frac{dy\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}\,.\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{yy} =$$

$$-D\frac{\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}\,.\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{y}$$

$$+\frac{dy\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}}{\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}+\frac{dy\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}}$$: ergo

facta substitutione proveniet

$$\frac{zz\,dz}{\sqrt{a+bzz+cz^4}} = -D\frac{\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}\,.\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{y\sqrt{2c}}$$

$$+\frac{dy\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}}{\sqrt{2c}\cdot\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}+\frac{dy\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{\sqrt{2c}\cdot\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}}.$$

Q. E. Inv.

VI. Ex his formulas alias magis patentes nancifcor, & ad canonicam perduco. Hanc ob rem fumo differentiam formulæ $z^r\sqrt{a+bzz+cz^4}$ hoc modo

$$Dz^r\sqrt{a+bzz+cz^4}=rz^{r-1}dz\sqrt{a+bzz+cz^4}+\frac{bz^{r+1}dz+2cz^{r+3}dz}{\sqrt{a+bzz+cz^4}}.$$

Formulas duas ad eamdem denominationem reduco, ut fiat

$$Dz^r\sqrt{a+bzz+cz^4}=\frac{raz^{r-1}dz+\overline{r+1}\,bz^{r+1}dz+\overline{r+2}\,.\,cz^{r+3}dz}{\sqrt{a+bzz+cz^4}}.$$

VII. Æquationem hanc primum ita difpono

$$(A)\ \frac{z^{r+3}dz}{\sqrt{a+bzz+cz^4}}=Dz^r\frac{\sqrt{a+bzz+cz^4}}{\overline{r+2}\,.\,c}$$

$$-(B)\ \frac{\overline{r+1}\,.\,bz^{r+1}dz}{\overline{r+2}\,.\,c\sqrt{a+bzz+cz^4}}-(C)\ \frac{raz^{r-1}dz}{\overline{r+2}\,.\,c\sqrt{a+bzz+cz^4}}.$$

Si $r=1$, formulæ B, C reducuntur ad canonicam vel ex N. III, & I, fi trinomium refolvi poffit in duo binomia realia, vel ex N. V, & II, fi refolutio binomii in trinomia realia fieri non poffit. Si $r=3$, formula B ex cafu fuperiore, formula C ex N. III, aut V ad canonicam perducitur. Si $r=5$, formulæ B, C ex duobus cafibus fuperioribus refolvuntur, atque ita dein-

deinceps in infinitum: ergo formula $\frac{z^{r+3}dz}{\sqrt{a+bzz+cz^4}}$ existente r numero positivo, & impari semper reducitur ad canonicam, atque integratur per arcus ellypticos, & hyperbolicos.

Scholium. Notum est omnibus analystis, formulas duas $\frac{zdz}{\sqrt{a+bzz+cz^4}}$, $\frac{dz}{z\sqrt{a+bzz+cz^4}}$ ad sui integrationem ad summum requirere quadraturas circuli, & hyperbolæ. Colligemus proinde, formulam superiorem $\frac{z^{r+3}dz}{\sqrt{a+bzz+cz^4}}$, existente r numero pari, & positivo integrari quadratis circulo, & hyperbola. Nam posito $r=o$, apparet, $\frac{z^3dz}{\sqrt{a+bzz+cz^4}}$ dependere a duabus suprapositis; facto $r=2$, formula $\frac{z^5dz}{\sqrt{a+bzz+cz^4}}$ dependet ab integratione duarum $\frac{z^3dz}{\sqrt{a+bzz+cz^4}}$, $\frac{zdz}{\sqrt{a+bzz+cz^4}}$; atque ita deinceps progressu satis manifesto.

VIII. Eamdem æquationem N. VI nova hac ratione dispono

$$(A)\ \frac{z^{r-1}dz}{\sqrt{a+bzz+cz^4}}=D\frac{z^r\sqrt{a+bzz+cz^4}}{ra}$$

$$-(B)\frac{\overline{r+1}.bz^{r+1}dz}{ra\sqrt{a+bzz+cz^4}}-(C)\frac{\overline{r+2}.cz^{r+3}dz}{ra\sqrt{a+bzz+cz^4}}.$$ Si $r=-1$,

$r = -1$, formula B, evanescit; formula autem C ex N. III, aut V ad canonicam reducitur. Si $r = -3$, formula B ex casu superiore, formula C ex N. III, aut V integratur. Si $r = -5$, ex duobus casibus superioribus formulæ B, C sunt in potestate, atque ita deinceps: Ergo formula $\frac{z^{r-1}dz}{\sqrt{a+bzz+cz^4}}$ est in potestate, si r est numerus impar, & negativus.

Scholium. Si statuas r æqualem numero pari, & negativo, patebit formulam $\frac{z^{r-1}dz}{\sqrt{a+bzz+cz^4}}$ ad sui integrationem non indigere, nisi quadraturis circuli, & hyperbolæ. Namque facto $r = -2$, $\frac{dz}{z^3\sqrt{a+bzz+cz^4}}$ pendebit a duabus $\frac{dz}{z\sqrt{a+bz^2+cz^4}}$, $\frac{zdz}{\sqrt{a+bzz+cz^4}}$; facto $r = -4$, $\frac{dz}{z^5\sqrt{a+bzz+cz^4}}$ pendebit a $\frac{dz}{z^3\sqrt{a+bzz+cz^4}}$, $\frac{dz}{z\sqrt{a+bzz+cz^4}}$; atque ita deinceps progressu satis manifesto.

IX. Ex his, quæ hactenus demonstrata sunt, manifestum est, formulam $\frac{z^t dz}{\sqrt{a+bzz+cz^4}}$ reduci ad canonicam, atque adeo integrari per arcus ellypticos & hyperbolicos, quotiescumque t sit numerus par vel positivus, vel negativus. Si vero t sit impar, non indiget nisi quadraturis circuli, & hyperbolæ.

X. Fran-

X. Transeo ad formulam $\frac{z^t dz \sqrt{f+gzz}}{\sqrt{p+qzz}}$, quæ nullo negotio reducitur ad superiorem. Nam facta multiplicatione, & divisione per $\sqrt{f+gzz}$ oritur

$$\frac{z^t dz \sqrt{f+gzz}}{\sqrt{p+qzz}} = \frac{fz^t dz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}} + \frac{gz^{t+2} dz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}$$

, quarum utraque in N. IX continetur. Itaque si t sit numerus par vel affirmativus, vel negativus integratur supposita rectificatione ellypsis, & hyperbolæ; si t sit impar, contenta est quadraturis circuli, & hyperbolæ.

XI. Idem dicendum est de formula $z^t dz \sqrt{a+bzz+cz^4}$. Fiat enim multiplicatio, & divisio per $\sqrt{a+bzz+cz^4}$; atque hæc obtinetur æquatio $z^t dz \sqrt{a+bzz+cz^4} =$

$$\frac{az^t dz}{\sqrt{a+bzz+cz^4}} + \frac{bz^{t+2} dz}{\sqrt{a+bzz+cz^4}} + \frac{cz^{t+4} dz}{\sqrt{a+bzz+cz^4}},$$

quæ tres formulæ spectant ad N. IX. Igitur si t sit numerus par vel positivus, vel negativus, $z^t dz \sqrt{a+bzz+cz^4}$ integratur per arcus ellypticos, & hyperbolicos; si t sit impar, per solas quadraturas circularem, & hyperbolicam.

XII. Luce clarius est, formulam

$z^t . \overline{a+bzz+cz^4}^e . dz \sqrt{a+bzz+cz^4}$ ad superiorem pertinere, si e sit numerus integer, & affirmativus; nam elato trinomio ad potestatem e, plures formulæ exorientur, quæ ad superiorem pertinebunt. Atqui formula exposita huic æquivalet

$z^t dz . \overline{a+bzz+cz^4}^{\frac{2e+1}{2}}$, in qua $2e+1$ erit numerus impar,

par, & positivus: ergo existente r numero impari, & positivo, formula $z^t dz . \overline{a+bzz+cz^4}^{\frac{r}{2}}$, integratur, si t sit par, per arcus ellypticos, & hyperbolicos; si t sit impar, per quadraturas circuli, & hyperbolæ.

Alio modo idem ostenditur. Formula

$$\frac{z^t . \overline{a+bzz+cz^4}^e . dz}{\sqrt{a+bzz+cz^4}}$$ convertitur in plures formulas, quæ ex N. IX integrantur, elevato trinomio ad potestatem e positivam, & integram: sed hæc æquivalet $z^t dz . \overline{a+bzz+cz^4}^{\frac{2e-1}{2}}$: ergo facto $2e-1=r$, qui necessario erit numerus impar, & positivus, habebimus formulam $z^t dz . \overline{a+bzz+cz^4}^{\frac{r}{2}}$, quæ integrabitur per arcus ellypticos & hyperbolicos, si t sit numerus par; per quadraturas circuli, & hyperbolæ, si t sit impar.

XIII. Simili ratione $\frac{z^t . \overline{f+gzz}^e . dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$ existente e numero integro, & positivo, elevato binomio ad potestatem e, convertitur in plures formulas, quæ integrantur ex N. X: sed exposita formula huic æquivalet $\frac{z^t dz . \overline{f+gzz}^{\frac{2e+1}{2}}}{\sqrt{p+qzz}}$, in qua $2e+1$ necessario est numerus impar: Ergo existente r numero impari, & positivo, formula $\frac{z^t dz . \overline{f+gzz}^{\frac{r}{2}}}{\sqrt{p+qzz}}$ obtinetur rectificatis ellypsi, & hyperbola, si t sit numerus par vel positivus, vel negativus; quadratis circulo, & hyperbola, si t sit impar.

XIV. Formula item $\frac{z^t dz . \overline{p+qzz}^e . \overline{f+gzz}^{\frac{r}{2}}}{\sqrt{p+qzz}}$ positis t, e, r numeris, qui supra definiti sunt, elevato binomio ad potestatem e, in plures mutatur, quæ integrationem recipiunt ex N. XIII: atqui ea æqualis est $z^t dz . \overline{p+gzz}^{\frac{2e-1}{2}} . \overline{f+gzz}^{\frac{r}{2}}$, in qua $2e-1$, est numerus impar, & positivus, qui fiat $= s$: Ergo $z^t dz . \overline{f+gzz}^{\frac{r}{2}} . \overline{p+qzz}^{\frac{s}{2}}$ rectificatis ellypsi & hyperbola integratur, si r, s sint numeri impares, & positivi, & t numerus par vel positivus, vel negativus. Si vero t sit impar vel positivus, vel negativus, formula per quadraturas circuli, & hyperbolæ absolvitur.

XV. Absolvam nunc formulam $\frac{z^t dz}{\overline{a+bzz+cz^4}^{\frac{r}{2}}}$, vel trinomium $a+bzz+cz^4$ possit resolvi in binomia realia, vel secus, existente t pari aut positivo aut negativo, & r impari, & positivo, quæ formula aliquanto est difficilior. Considero primum casus, in quibus aut $t > r+1$, aut $t = r+1$, aut $t = r-1$. Hanc ob rem formulam ita dispono $\frac{z^{t-r-1} . zdz . z^r}{\overline{a+bzz+cz^4}^{\frac{r}{2}}}$. Statuo $\frac{a+bzz+cz^4}{zz} = uu$, ut formula evadat $\frac{z^{t-r-1} . zdz}{u^r}$: atqui $z^4 + \frac{b-u^2}{c} zz = \frac{-a}{c}$: Igitur

$$z^2 = \frac{-b+uu}{2c} + \sqrt{\frac{\overline{b-u^2}^2}{4cc} - \frac{a}{c}} : \text{Ergo}$$

$$z\,dz = \frac{u\,du}{2c} - \frac{u\,du.\overline{b-uu}}{4cc\sqrt{\frac{\overline{b-u^2}^2}{4cc} - \frac{a}{c}}}.$$ Demum

$$\frac{z^t dz}{\overline{a+bzz+cz^4}^{\frac{r}{2}}} = \frac{du}{2cu^{r-1}} - \frac{du.\overline{b-u^2}}{4cc\sqrt{\frac{\overline{b-uu}^2}{4cc} - \frac{a}{c}}.u^{r-1}}$$

ducta in $\overline{\frac{-b+u^2}{2c} + \sqrt{\frac{\overline{b-u^2}^2}{4cc} - \frac{a}{c}}}^{\frac{t-r-1}{2}}$.

Quoniam tam t, quam $r+1$ supponitur par, evidens est numerum $t-r-1$ esse parem: Ergo $\frac{t-r-1}{2}$ erit numerus integer. Itaque si sit t aut $>$, aut $= r+1$; elevato binomio ad potestatem integram $\frac{t-r-1}{2}$, factaque multiplicatione, plures exorientur formulæ, quæ aut erunt integrabiles, aut pertinebunt ad N. IX, & XI, & indigebunt sectionum conicarum rectificatione.

Si $t = r-1$ nostra formula in hanc mutatur

$$\frac{z^{r-1} dz}{\overline{a+bzz+cz^4}^{\frac{r}{2}}} = \frac{du}{2cu^{r-1}\sqrt{\frac{\overline{b-uu}^2}{4cc} - \frac{a}{c}}}$$ quæ ex

N. IX per rectificationem sectionum integratur.

XVI. Ad reliquos casus absolvendos pone $z = \frac{1}{u}$, & habebis $\frac{z^t dz}{\overline{a+bz^2+cz^4}^{\frac{r}{2}}} = \frac{-u^{-t+2r-2}du}{\overline{au^4+bu^2+c}^{\frac{r}{2}}}$. Hæc formu-

la

la ex numero superiore integratur si $-t+2r-2$ sit aut $>$, aut $=r+1$, sive si t sit aut $<$, aut $=r-3$, quæ hypothesis includit etiam t negativam: sed hi casus unice reliqui erant ad plenissime integrandam formulam.

Si ponerem $-t+2r-2=r-1$, qui casus item ex numero superiori absolvitur, iterum provenit $t=r-1$, qui pariter eodem loco absolutus est.

XVII. Ex duobus numeris superioribus constat, formulam $\frac{z^t dz}{\overline{f+gzz}^{\frac{r}{2}} \cdot \overline{p+qzz}^{\frac{r}{2}}}$ egere rectificatione conicarum sectionum, si r sit numerus impar, & t par vel positivus, vel negativus: Ergo $\frac{\overline{f+gzz}^e \cdot z^t dz}{\overline{f+gz^2}^{\frac{r}{2}} \cdot \overline{p+qzz}^{\frac{r}{2}}}$ similiter integrabitur, si e sit numerus integer, & positivus. Nam elevato binomio ad potestatem e mutatur formula in plures, quæ ad superiorem spectant. Atqui formula æquivalet huic $\frac{\overline{f+gzz}^{\frac{2e-r}{2}} dz}{\overline{p+qz^2}^{\frac{r}{2}}}$, in qua $2e-r$ est semper numerus impar. Pone $2e-r=s$, si $2e>r$; si $2e<r$, fac $2e-r=-s$. Provenient formulæ duæ $\frac{z^t dz \cdot \overline{f+gzz}^{\frac{s}{2}}}{\overline{p+qzz}^{\frac{s}{2}}}$, $\frac{z^t dz}{\overline{f+gzz}^{\frac{s}{2}} \cdot \overline{p+qzz}^{\frac{r}{2}}}$, quæ rectificatis sectionibus conicis integrantur.

XVIII. Omnes formulæ, quæ hactenus rectificatis sectionibus conicis integratæ sunt, ad has duas reducuntur

$$z^t dz \cdot \overline{a+bzz+cz^4}^{\frac{r}{2}}, \quad z^t dz \cdot \overline{f+gzz}^{\frac{s}{2}} \cdot \overline{p+gzz}^{\frac{r}{2}},$$

in quibus t est numerus par vel positivus, vel negativus; r, s sunt impares pariter vel negativi, vel positivi.

 Scho-

Scholium. Formulæ istæ duæ, quæ complectuntur omnes illas, quæ tractatæ sunt a N. XII usque ad N. XVII, si vel alteruter, vel uterque ex numeris r, s sit par, aut t sit impar, sectionum conicarum rectificatione non indigebunt, sed integrabiles erunt, vel algebraice, vel per notas quadraturas circuli, & hyperbolæ.

XIX. Ex numero superiore facile deducimus, formulas

$$x^{\frac{u}{2}} dx . \overline{a+bx+cx^2}^{\frac{r}{2}}, \quad x^{\frac{u}{2}} dx . \overline{f+gx^2}^{\frac{s}{2}} . \overline{p+qx^2}^{\frac{r}{2}}$$

integrari rectificatis sectionibus conicis, si numeri u, s, r sint omnes impares vel positivi, vel negativi. Nam fiat $x = zz$, & orientur

$$2z^{u+1} dz . \overline{a+bzz+zz^4}^{\frac{r}{2}}, \quad 2z^{u+1} dz . \overline{f+gzz}^{\frac{s}{2}} . \overline{p+qzz}^{\frac{r}{2}},$$

in quibus $u+1$ est par: Ergo istæ coincidunt cum formulis numeri superioris.

XX. Formulæ duæ $x^t dx . \overline{k+bx}^{\frac{u}{2}} . \overline{a+bx+cx^2}^{\frac{r}{2}}$, $x^t dx . \overline{k+bx}^{\frac{u}{2}} . \overline{f+gx}^{\frac{s}{2}} . \overline{p+qx}^{\frac{r}{2}}$, si t sit numerus integer, & positivus; u, s, r numeri impares vel positivi, vel negativi, obtinentur suppositis rectificationibus ellypsis & hyperbolæ. Fiat enim $k+bx = by$ & orietur

$$b^{\frac{u}{2}} . \overline{y - \frac{k}{b}}^{t} . y^{\frac{u}{2}} dy . \overline{a - \frac{bk}{b} + \frac{ck^2}{b^2} + by - \frac{2ck}{b}y + cy^2}^{\frac{r}{2}}$$

$$b^{\frac{u}{2}} . \overline{y - \frac{k}{b}}^{t} . y^{\frac{u}{2}} dy . \overline{f - \frac{gk}{b} + gy^2}^{\frac{s}{2}} . \overline{p - \frac{qk}{b} + gy}^{\frac{r}{2}},$$ quæ elevato

bino-

binomio ad potestatem integram, & positivam t, factaque multiplicatione plures orientur termini, qui omnes integrabuntur per numerum superiorem.

Scholium. Adverte, tria binomia $k+bx$, $f+gx$, $\overline{p+qx}$ inæqualia esse oportere; secus formula non egeret rectificatione sectionum conicarum.

Corollarium. Si $u=s=r$, formula hanc formam indueret $x^t dx \,.\, \overline{a+bx+cx^2+ex^3}^{\frac{r}{2}}$, quæ hanc continet $x^t dx \sqrt{a+bx+cx^2+ex^3}$: nam quadrinomium $a+bx+cx^2+ex^3$ semper habet factorem realem hujus formæ $k+bx$.

XXI. Quamquam, existente t numero positivo, de formula $\frac{dx}{x^t\sqrt{a+bx+cx^2+ex^3}}$ nihil possumus pronunciare: tamen, si ambæ b, $c=o$, facile statuemus, quandonam formula contenta sit notis quadraturis circuli, & hyperbolæ, quandonam poscat rectificationes hyperbolæ, & ellypsis. Hanc ob rem præmitto primo, formulam $\frac{dx}{x\sqrt{a+ex^3}}$ integrari per quadraturam circuli, aut hyperbolæ, quod tibi constabit, si utaris substitutione $\sqrt{a+ex^3}=y$. Præmitto deinde, formulas $\frac{xdx}{\sqrt{a+ex^3}}$, $\frac{dx}{\sqrt{a+ex^3}}$ postulare rectificationem ellypsis, & hyperbolæ, ut probatum est numero superiore.

His præmissis sumo differentiale formulæ $\frac{\sqrt{a+ex^3}}{x^q}$, quæ est hujusmodi

$$D\frac{\sqrt{a+ex^3}}{x^q}=\frac{-q\,dx\sqrt{a+ex^3}}{x^{q+1}}+\frac{3e\,dx}{2x^{q-2}\sqrt{a+ex^3}}$$

sive facta opportuna reductione

$$D\frac{\sqrt{a+ex^3}}{x^q}=\frac{-qa\,dx}{x^{q+1}\sqrt{a+ex^3}}\quad\frac{-\overline{2q+3}.e\,dx}{2x^{q-2}\sqrt{a+ex^3}}:$$

XXII. Formulam inventam ita dispono

$$(A)\frac{dx}{x^{q+1}\sqrt{a+ex^3}}=-D\frac{\sqrt{a+ex^3}}{qax^q}(B)\frac{-\overline{2q+3}.e\,dx}{2qax^{q-2}\sqrt{a+ex^3}}$$

ex qua constat formulam A dependere a B. Si $q=1$, formula B poscit rectificationes; si $q=4$, B integratur ex casu primo; si $q=7$, ex secundo; atque ita deinceps: Ergo si q sit numerus ex hac serie 1, 4, 7, 10, 13 & cæ. formula A poscit rectificationes conicarum sectionum.

Si q sit $=2$, formula B pariter indiget rectificatione sectionum conicarum: Ergo etiam formula A; si $q=5$, formula B integratur ex primo casu; si $q=8$, ex secundo; atque ita deinceps. Ergo formula A integratur per rectificationem sectionum conicarum, si q sit numerus ex serie

2, 5, 8, 11, 14 & cæ.

Demum si $q=3$, formula B solas quadraturas requirit: Ergo etiam formula A; si $q=6$, formula B integrarur ex primo casu; si $q=9$, ex secundo; atque ita deinceps: Ergo per quadraturas circuli, & hyperbolæ integratur formula A, si q in hac serie contineatur

3, 6, 9, 12; 15 & cæ.

Quare divisis omnibus numeris in tres series

1 4 7 10 13 & cæ. cui est terminus generalis $=3n-2$
2 5 8 11 14 & cæ. cujus terminus generalis $=3n-1$
3 6 9 12 15 & cæ. cui convenit terminus generalis $=3n$

formula $\frac{dx}{x^t\sqrt{a+ex^3}}$ per solas quadraturas circuli & hyperbo-

læ

læ integratur, si t contineatur in prima serie, sive posito n numero integro sit $= 3n - 2$; per rectificationem hyperbolæ & ellypsis, si t contineatur aut in secunda, aut in tertia serie, sive sit aut $t = 3n - 1$, aut $t = 3n$.

XXIII. Ex superiore facile determinatur formula $\frac{dx\sqrt{a+ex^3}}{x^t}$: nam facta multiplicatione per $\sqrt{a+ex^3}$ oritur

$$\frac{dx\sqrt{a+ex^3}}{x^t} = \frac{dx.\overline{a+ex^3}}{x^t\sqrt{a+ex^3}} = \frac{adx}{x^t\sqrt{a+ex^3}} + \frac{edx}{x^{t-3}\sqrt{a+ex^3}},$$

quarum utraque est in potestate. Si existente n quolibet numero integro, & positivo, fuerit aut $t = 3n$, aut $t = 3n - 1$, utraque formula requirit rectificationem ellypsis, & hyperbolæ; si vero $= 3n - 2$, contenta est notis quadraturis circuli, & hyperbolæ.

Alio modo. Si æquationem inventam N. XXI opportune disponas, hanc obtinebis

$$\frac{dx\sqrt{a+ex^3}}{x^{q+1}} = -D\frac{\sqrt{a+ex^3}}{qx^q} + \frac{3edx}{2qx^{q-2}\sqrt{a+ex^3}}$$ sive

facta $q + 1 = t$

$$\frac{dx\sqrt{a+ex^3}}{x^t} = -D\frac{\sqrt{a+ex^3}}{\overline{t-1}.x^{t-1}} + \frac{3edx}{2.\overline{t-1}.x^{t-3}\sqrt{a+ex^3}},$$

quarum ultima ex superiori numero est in potestate. Excipias tamen velim casum, in quo $t = 1$: nam formulæ duæ dividuntur per o. Verum formula $\frac{dx\sqrt{a+ex^3}}{x}$ integratur per quadraturam circuli, & hyperbolæ, ut cognosces si utaris substitutione $\sqrt{a+ex^3} = y$.

XXIV.

XXIV. Hinc colligimus integrationem formulæ $\frac{dx \cdot (a+ex^3)^{\frac{r}{2}}}{x^t}$, existente r numero impari, & positivo, non superare rectificationem sectionum conicarum. Namque ea in hanc formam mutari potest $\frac{(a+ex^3)^{\frac{r+1}{2}}\, dx}{x^t\sqrt{a+ex^3}}$, in qua $r+1$ debet esse numerus par, & $\frac{r+1}{2}$ numerus integer: Ergo elevato binomio ad potestatem positivam, & integram, formula in plures dividetur, quæ ex N. XX, & XXII, sunt in potestate.

XXV. Nihil reliquum est, nisi ut integremus formulam $\frac{dx}{x^t \cdot (a+ex^3)^{\frac{r}{2}}}$. Hanc ob rem sumo differentiale quantitatis $\frac{1}{x^{t+2} \cdot (a+ex^3)^{\frac{r-2}{2}}}$, & invenio

$$D\frac{1}{x^{t+2} \cdot (a+ex^3)^{\frac{r-2}{2}}} = \frac{-(t-2) \cdot dx}{x^{t+3} \cdot (a+ex^3)^{\frac{r-2}{2}}} - \frac{3 \cdot (r-2) \cdot dx}{2x^t \cdot (a+ex^3)^{\frac{r}{2}}},$$ sive

$$\text{(A)}\quad \frac{dx}{x^t \cdot (a+ex^3)^{\frac{r}{2}}} = \frac{-2}{3 \cdot (r-2)} D\frac{1}{x^{t+2} \cdot (a+ex^3)^{\frac{r-2}{2}}} -$$

$$\text{(B)}\quad \frac{2 \cdot (t+2) \cdot dx}{3 \cdot (r-2) \cdot x^{t+3} \cdot (a+ex^3)^{\frac{r-2}{2}}},$$ ex qua æquatione constat,

formu-

formulam A dependere a B. Si $r=3$, formula B est in potestate: ergo etiam formula A; si $r=5$, formula B ex casu primo nota est, adeoque fiet nota formula A; si $r=7$, B ex casu secundo innotescit, igitur & formula A. Quare progressu in infinitum producto constat, formulam A esse in potestate, neque ad sui integrationem requirere plus quam sectionum conicarum rectificationes.

XXVI. Quapropter generatim formula $x^t dx . \overline{a+ex^3}^{\frac{r}{2}}$, si t sit numerus integer, r præterea impar, & uterque vel positivus, vel negativus, vel est algebraice integrabilis, vel pendet a quadratura circuli, & hyperbolæ, vel certe per rectificationem hyperbolæ & ellypsis integratur.

Scholium. Quæ dicta sunt, satis ostendunt, formulam non indigere rectificatione sectionum conicarum, si $t=3n+2$, etiamsi n positiva sit: immo in hac hypotesi formula est algebraice integrabilis, quod palam faciet substitutio $\overline{a+ex^3}^{\frac{r}{2}}=y$.

XXVII. Si t sit numerus integer & positivus, u, s, r numeri integri, & impares vel positivi vel negativi, formulæ

$$\frac{dx . \overline{k+bx}^{\frac{u}{2}} . \overline{a+bx+cx^2}^{\frac{r}{2}}}{x^{t+2+\frac{u+2r}{2}}}$$

$$\frac{dx . \overline{k+bx}^{\frac{u}{2}} . \overline{f+gx}^{\frac{s}{2}} . \overline{p+qx}^{\frac{r}{2}}}{x^{t+2+\frac{u+s+r}{2}}}$$ facillime reducuntur

ad N. XX. Fiat enim $x=\frac{1}{y}$, & orientur

$$-y^t dy . \overline{ky+b}^{\frac{u}{2}} . \overline{ay^2+by+c}^{\frac{r}{2}}$$

$$-y^t dy . \overline{ky+b}^{\frac{u}{2}} . \overline{fy+g}^{\frac{s}{2}} . \overline{py+q}^{\frac{r}{2}}$$, quæ ex N. XX integrationem accipiunt.

Corollarium. In formulis superioribus, quum t debeat esse positivus numerus, & integer, evidens est $2t+4$, debere esse numerum parem, neque posse esse < 4; cæterum t foret negativus. Quare si deinceps per speciem t intelligamus numerum positivum parem, neque minorem quam 4, formulæ sequentes erunt in potestate, existentibus u, r, s imparibus vel positivis, vel negativis

$$\frac{dx.\,\overline{k+bx}^{\frac{u}{2}}.\,\overline{a+bx+cx^2}^{\frac{r}{2}}}{x^{\frac{t+u+2r}{2}}}$$

$$\frac{dx.\,\overline{k+bx}^{\frac{u}{2}}.\,\overline{f+gx}^{\frac{s}{2}}.\,\overline{p+qx}^{\frac{r}{2}}}{x^{\frac{t+u+s+r}{2}}}.$$

XXVIII. Iisdem suppositis

$$\frac{dx.\,\overline{k+bx}^{\frac{u}{2}}.\,\overline{a+bx+cx^2}^{\frac{r}{2}}}{\overline{m+nx}^{\frac{t+u+2r}{2}}}$$

$$\frac{dx.\,\overline{k+bx}^{\frac{u}{2}}.\,\overline{f+gx}^{\frac{s}{2}}.\,\overline{p+qx}^{\frac{r}{2}}}{\overline{m+nx}^{\frac{t+u+s+r}{2}}}$$ nullo negotio reducuntur ad superiores facta substitutione $m+nx=ny$.

XXIX. Si t sit numerus integer vel positivus, vel negativus, r ut supra, formula $\frac{dx.\,\overline{a+ex^3}^{\frac{r}{2}}}{x^{t+2+\frac{3r}{2}}}$ per substitutionem $x=\frac{1}{y}$ in hanc mutatur $-y^t dy.\,\overline{ay^3+e}^{\frac{r}{2}}$, quæ ex numero

mero

mero XXVI semper integratur vel absolute, vel per notas quadraturas, vel sectionum conicarum rectificationes.

Corollarium. Quoniam $2t+4$ in formula superiore debet esse numerus par vel positivus, vel negativus, si deinceps per t intelligamus hunc numerum, formula $\frac{dx.\overline{a+ex^3}^{\frac{r}{2}}}{x^{\frac{t+3r}{2}}}$ semper erit in potestate.

XXX. Quare iisdem positis, formula $\frac{dx.\overline{a+ex^3}^{\frac{r}{2}}}{\overline{m+nx}^{\frac{t+3r}{2}}}$ est in potestate; quia nullo negotio reducitur ad superiorem ope- substitutionis $m+nx=ny$.

Scholium. Aliqua adnotanda sunt maximi momenti in formulis, quæ continentur N. XXVII, XXVIII. Generatim formulæ $dx.\overline{m+nx}^{\frac{\varphi}{2}}.\overline{k+bx}^{\frac{u}{2}}.\overline{a+bx+cx^2}^{\frac{r}{2}}$ $dx.\overline{m+nx}^{\frac{\varphi}{2}}.\overline{k+bx}^{\frac{u}{2}}.\overline{f+gx}^{\frac{s}{2}}.\overline{p+qx}^{\frac{r}{2}}$ in quibus potest etiam $m=o$, & $n=1$, integrari non possunt etiam advocatis rectificationibus sectionum conicarum, si φ, u, s, r sint numeri impares, & positivi. Namque secunda formula collata cum formulis N. XXVII, aut XXVIII daret $t+u+r+s=-\varphi$. Ergo $t=-\varphi-u-r-s$, atque adeo t esset negativa: idem dic si faciamus comparationem in prima formula; atqui existente t negativa formulæ non sunt in potestate: ergo neque propositæ. Verum formula

$dx.\overline{m+nx}^{\frac{\varphi}{2}}.\overline{a+ex^3}^{\frac{r}{2}}$ integratur ex N. XXX, nam ea formula, licet t sit negativus numerus, dummodo integer, absolvitur.

Ut formula $\dfrac{dx\,.\,\overline{k+bx}^{\frac{u}{2}}\,.\,\overline{f+gx}^{\frac{s}{2}}\,.\,\overline{p+qx}^{\frac{r}{2}}}{\overline{m+nx}^{\frac{\phi}{2}}}$ sit in poteſtate, quum exponentes omnes ſunt poſitivi, neceſſe eſt, ut ϕ ſuperet ſummam aliorum exponentium u, s, r ſaltem per 4 unitates: nam erit $\phi = t + u + s + r$: ergo $\phi - u - s - r = t$, ſed t non debet < 4: ergo. Idem dicas de alia formula $\dfrac{dx\,.\,\overline{k+bx}^{\frac{u}{2}}\,.\,\overline{a+bx+ex^2}^{\frac{r}{2}}}{\overline{m+nx}^{\frac{\phi}{2}}}$. Verum hæc conditio non requiritur in formula $\dfrac{dx\,.\,\overline{a+cx^3}^{\frac{r}{2}}}{\overline{m+nx}^{\frac{\phi}{2}}}$, nam hæc ſemper ad integrationem perducitur ſaltem per rectificationem conicarum ſectionum.

De formula $\dfrac{dx\,.\,\overline{f+gx}^{\frac{s}{2}}\,.\,\overline{p+qx}^{\frac{r}{2}}}{\overline{k+bx}^{\frac{u}{2}}\,.\,\overline{m+nx}^{\frac{\phi}{2}}}$ pronuncia, eam eſſe in poteſtate ſi $u + \phi - s - r$ non ſit < 4. Idem dicas de formula $\dfrac{dx\,.\,\overline{a+bx+cx^2}^{\frac{r}{2}}}{\overline{k+bx}^{\frac{u}{2}}\,.\,\overline{m+nx}^{\frac{\phi}{2}}}$ facta $s = r$. Immo etiam de formula $\dfrac{dx\,.\,\overline{k+bx}^{\frac{u}{2}}\,.\,\overline{m+nx}^{\frac{\phi}{2}}}{\overline{a+bx+cx^2}^{\frac{r}{2}}}$, ſi $2r$ ſuperet $u + \phi$

falsaltem per quatuor unitates. Itaque

$$\frac{dx . \overline{a + bx + cx^2}^{\frac{r}{2}}}{\overline{f + gx + hx^2}^{\frac{s}{2}}}$$, si $2s$ superet $2r$ saltem per quatuor unitates, sive s superet r per unitates duas, absolvitur, quotiescumque alterutrum ex trinomiis sit resolubile in factores reales. Verum si neutrum hanc resolutionem admittat, nondum liquet, qua ratione formula integrari possit.

Item formula $\frac{dx . \overline{k + hx}^{\frac{u}{2}}}{\overline{m + nx}^{\frac{\varphi}{2}} . \overline{f + gx^2}^{\frac{s}{2}} . \overline{p + qx^2}^{\frac{r}{2}}}$ erit in potestate si $\varphi + s + r$ superet u saltem per quatuor unitates; quod dicendum est etiamsi $r = s$, & habeatur trinomium non resolubile in factores reales. Formula autem $\frac{dx . \overline{m + nx}^{\frac{\varphi}{2}}}{\overline{a + ex^3}^{\frac{r}{2}}}$ semper absolvitur.

Demum formula $\frac{dx.}{\overline{m + nx}^{\frac{\varphi}{2}} . \overline{k + hx}^{\frac{u}{2}} . \overline{f + gx}^{\frac{s}{2}} . \overline{p + qx}^{\frac{r}{2}}}$ semper absolvitur. Verum formula

$$\frac{dx}{\overline{a + bx + cx^2}^{\frac{r}{2}} . \overline{f + gx + hx^2}^{\frac{s}{2}}}$$ conficietur, si alterutrum ex trinomiis sit resolubile in factores reales. Quod si neutrum resolubile sit, nondum constat, quo pacto formula integretur.

Quæ dicta sunt hactenus, hanc regulam suppeditant œcumenicam. Quotiescumque in differentiali formula adsint tantum quatuor binomia primi ordinis elata ad potestatem imparem divisam per 2, & omisso hoc divisore 2 summa exponentium binomiorum, quæ sunt in divisore, superet saltem quatuor unitatibus

tibus ſummam ſimilium exponentium in numeratore, methodus ſuppetit, qua ſaltem per rectificationem ſectionum conicarum ad integrationem perducatur. Si deſit hæc conditio; de integratione nihil licet pronunciare. Idem dicas ſi adſit trinomium ſecundi ordinis, quod tamquam duo binomia ſpectandum erit, adeoque ejus exponens bis erit accipiendum. Verum ſi duo trinomia adſint, quorum neutrum in binomia realia reſolvi poſſit, nondum conſtat, quo pacto formula poſſit integrari. Quod ſi adſint duo binomia unum primi ordinis, alterum tertii, ſemper formula aliqua ratione ad integrationem perducitur.

XXXI. Reducamus nunc ad arcus ellypticos, & hyperbolicos formulam $\frac{dx}{\overline{a+bx+cx^2}^{\,t}\cdot\overline{g+\frac{b}{a+bx+cx^2}}^{\,\frac{s}{2}}}$, in qua t eſt numerus integer, s præterea impar. Utere ſubſtitutione $\frac{a+bx+cx^2}{b}=\frac{z}{A}$: quantitas A determinabitur in progreſſu, prout libuerit. Formula ſtatim in hanc mutatur $\frac{A^t\,dx}{b^t\,z^{t-\frac{s}{2}}\cdot\overline{gz+b}^{\,\frac{s}{2}}}$. Ex æquatione ſubſtitutionis habebimus $x^2+\frac{bx}{c}=\frac{bz}{cA}-\frac{a}{c}$; ergo

$$x=\frac{-b}{2c}+\sqrt{\frac{bz}{cA}+\frac{bb}{4cc}-\frac{a}{c}},$$ & facta $A=\frac{b}{c}$ erit

$$x=\frac{-b}{2c}+\sqrt{z+\frac{bb}{4cc}-\frac{a}{c}}$$: igitur

$$dx=\frac{dz}{2\sqrt{z+\frac{bb}{4cc}-\frac{a}{c}}}$$: quo valore ſubſtituto in formula

$$\frac{dx}{c^{t} \cdot z^{\frac{2t-s}{2}} \cdot \overline{gz+b}^{\frac{s}{2}}}$$, quæ resultat ex ea, quæ paullo ante inventa est, substituto valore A, habebimus

$$\frac{dz}{2c^{t} \cdot z^{\frac{2t-s}{2}} \cdot \sqrt{z+\frac{bb}{4cc}-\frac{a}{c}} \cdot \overline{gz+b}^{\frac{s}{2}}}$$, quæ ex N. XIX, suppositis rectificationibus sectionum conicarum, semper integratur.

XXXII. Methodus superioris numeri palam docet, etiam formulam $$\frac{x^{m}dx}{\overline{a+bx+cx^{2}}^{t} \cdot \overline{g+\frac{b}{a+bx+cx^{2}}}^{\frac{s}{2}}}$$, existente m numero positivo & integro, per arcus sectionum conicarum integrari. Nam factis iisdem substitutionibus orietur

$$\frac{dz \cdot \overline{\frac{-b}{2c}+\sqrt{z+\frac{bb}{4cc}-\frac{a}{c}}}^{m}}{2c^{t} \cdot z^{\frac{2t-s}{2}} \cdot \overline{z+\frac{bb}{4cc}-\frac{a}{c}}^{\frac{1}{2}} \cdot \overline{gz+b}^{\frac{s}{2}}}$$. Si elevetur binomium radicale ad potestatem integram, & positivam m, formula in plures dividetur, quarum aliquæ non indigebunt rectificatione ellypsis, & hyperbolæ, aliæ per hanc rectificationem ex N. XIX integrabuntur.

Si $b=o$, etiamsi m sit numerus negativus, dummodo integer, formula continetur N. XIX.

XXXIII. Formulæ superiores inserviunt integrandis formulis

$$\frac{dx}{\overline{a+bx+cx^{2}}^{\frac{r}{2}} \cdot \overline{f+gx+bx^{2}}^{\frac{s}{2}}}$$

$x^{m}dx$

$$\frac{x^m dx}{\overline{a+bx+cx^2}^{\frac{r}{2}} \cdot \overline{f+gx+bx^2}^{\frac{s}{2}}}$$

, existente m numero integro, & positivo, & r, s numeris imparibus vel positivis, vel negativis, quando $bb=cg$. Nam secunda formula, quæ in primam desinit, quum $m=o$, ita disponi potest

$$\frac{x^m dx}{\overline{a+bx+cx^2}^{\frac{r+s}{2}}} \cdot \left(\frac{f+gx+bx^2}{a+bx+cx^2}\right)^{\frac{s}{2}}$$

sive

$$\frac{x^m dx}{\overline{a+bx+cx^2}^{\frac{r+s}{2}} \cdot \overline{\frac{\frac{b}{c}+g-\frac{bb}{c}\cdot x+f-\frac{ab}{c}}{a+bx+cx^2}}^{\frac{s}{2}}}$$

, quæ, si $gc=bb$, ex duobus numeris superioribus recepit integrationem.

Protuli hujusmodi formulæ reductionem ad methodum, qua deinceps utemur, indicandam. Nam formula multo facilius reducitur. Ita enim disponatur

$$\frac{x^m dx}{c^{\frac{r}{2}} \cdot \overline{\frac{a}{c}+\frac{b}{c}x+x^2}^{\frac{r}{2}} \cdot b^{\frac{s}{2}} \cdot \overline{\frac{f}{b}+\frac{g}{b}x+x^2}^{\frac{s}{2}}}$$

. Fiat $x+\frac{b}{2c}=x+\frac{g}{2b}=z$. Igitur formula in hanc mutabitur

$$\frac{dz \cdot \overline{z-\frac{b}{2c}}^{m}}{c^{\frac{r}{2}} \cdot \overline{\frac{a}{c}-\frac{bb}{4cc}+zz}^{\frac{r}{2}} \cdot b^{\frac{s}{2}} \cdot \overline{\frac{f}{b}-\frac{gg}{4bb}+zz}^{\frac{s}{2}}}$$

, sive

dz.

$$\frac{dz \cdot \overline{z - \frac{b}{2c}}^{m}}{\overline{a - \frac{bb}{4c} + czz}^{\frac{r}{2}} \cdot \overline{f - \frac{gg}{4b} + bzz}^{\frac{s}{2}}}$$, quæ, elevato binomio ad poteſtatem m, plures formulas ſuppeditat pertinentes ad N. XIV.

XXXIV. Progredior ad formulam magis compoſitam, & difficilem $\frac{dx}{\overline{a + bx + cx^2}^{\frac{t}{2}} \cdot \overline{g + \frac{kx + h}{a + bx + cx^2}}^{\frac{s}{2}}}$, in qua t eſt numerus integer, s præterea impar. Utar ſubſtitutione $\frac{a + bx + cx^2}{kx + h} = \frac{z}{A}$, in qua A eſt quantitas determinanda in operationis progreſſu. Formula autem in hanc mutatur $\frac{A^t dx}{z^{\frac{2t - s}{2}} \cdot \overline{kx + h}^{\frac{t}{2}} \cdot \overline{gz + A}^{\frac{s}{2}}}$. Ab hac formula ut x arceamus, æquationem ſubſtitutionis ita diſtribue $x^2 + \overline{\frac{b}{c} - \frac{kz}{cA}} \cdot x = \frac{-a}{c} + \frac{hz}{cA}$, & reſolutione effecta

$$x = \frac{-b}{2c} + \frac{kz}{2cA} + \sqrt{\frac{bb}{4cc} - \frac{2bkz}{4ccA} + \frac{k^2zz}{4ccA^2} \atop \frac{-a}{c} + \frac{hz}{A}}$$

. Ut formula fiat ſimplicior, pone $A = \frac{k}{2c}$, & orietur

$x = \frac{-b}{2c} + z + \sqrt{\frac{bb}{4cc} - \frac{a}{c} - \frac{bz}{c} + \frac{2b}{k}z + zz}$. Ad faciliorem efficiendam analysim statuo $\frac{bb}{4cc} - \frac{a}{c} = m$, $\frac{2b}{k} - \frac{b}{c} = 2n$. Igitur $x = \frac{-b}{2c} + z + \sqrt{m + 2nz + zz}$: ergo sumptis differentiis $dx = dz + \frac{n\,dz + z\,dz}{\sqrt{m+2nz+zz}} = \frac{\overline{n+z+\sqrt{m+2nz+zz}}.dz}{\sqrt{m+2nz+zz}}$. Item $kx + b = \frac{-kb}{2c} + b + kz + k\sqrt{m+2nz+zz}$: atqui $\frac{-kb}{2c} + b = kn$: igitur

$$\frac{dx}{\overline{kx+b}^{\,t}} = \frac{dz}{k^t . \sqrt{m+2nz+zz} . \overline{n+z+\sqrt{m+2nz+zz}}^{\,t-1}}.$$

Quapropter proposita formula in hanc vertitur

$$\frac{dz}{2^t c^t . z^{\frac{2t-s}{2}} . \overline{gz + \frac{k}{2c}}^{\frac{s}{2}} \sqrt{m+2nz+zz} . \overline{n+z+\sqrt{m+2nz+zz}}^{\,t-1}}.$$

XXXV. Si $t = 1$, formula superior evadit multo simplicior, nempe $\frac{dz}{2c . z^{\frac{2-s}{2}} . \overline{gz + \frac{k}{2c}}^{\frac{s}{2}} . \sqrt{m+2nz+zz}}$. Hæc, quicumque sit numerus s vel positivus, vel negativus, continetur semper in N. XXVII; atque adeo integratur per rectificationes conicarum sectionum.

XXXVI. Si ponas $t = o$, formula in hanc mutabitur

$z^{\frac{s}{2}} dz.$

$\frac{z^{\frac{s}{2}} dz \cdot \overline{n+z+\sqrt{m+2nz+zz}}}{\overline{gz+\frac{k}{2c}}^{\frac{s}{2}} \cdot \sqrt{m+2nz+zz}}$, quæ facta multiplicatione in tres hasce transformatur

$$\frac{nz^{\frac{s}{2}} dz}{\overline{gz+\frac{k}{2c}}^{\frac{s}{2}} \cdot \sqrt{m+2nz+zz}} + \frac{z^{\frac{2+s}{2}} dz}{\overline{gz+\frac{k}{2c}}^{\frac{s}{2}} \cdot \sqrt{m+2nz+zz}} +$$

$\frac{z^{\frac{s}{2}} dz}{\overline{gz+\frac{k}{2c}}^{\frac{s}{2}}}$. Harum tertia non indiget rectificationibus sectionum conicarum, sed duæ primæ ne his quidem concessis integrari possunt. Nam, neglecto divisore 2, summa exponentium in denominatore non est major saltem quatuor unitatibus summa exponentium in numeratore.

Idem prorsus reperies, si t sit numerus integer quidem, sed negativus: nam translatis factoribus, prout opus est, in numeratorem, ut exponentes positivi fiant, elevatoque multinomio ad potestatem positivam $-t+1$, divisaque formula in plures, aliquæ exorientur formulæ, in quibus summa exponentium denominatoris, omisso divisore 2, non superat per quatuor unitates summam exponentium numeratoris: quæ formulæ quomodo integrentur, adhuc ignotum est.

XXXVII. Si vero t fuerit numerus integer positivus, & unitate major, præparare oportet formulam, eam multiplicando, ac dividendo per $\overline{n+z-\sqrt{m+2nz+zz}}^{\,t-1}$, ut hanc formam accipiat

$$\frac{dz\,.\,\overline{n+z-\sqrt{m+2nz+zz}}^{\,t-1}}{2^t\,.\,c^t\,.\,\overline{nn-m}^{\,t-1}\,.\,z^{\frac{2t-s}{2}}\,.\,\overline{gz+\frac{k}{2c}}^{\,\frac{s}{2}}\,.\,\sqrt{m+2nz+zz}}\,.$$

Sit primo $t=2$, & formula in has tres tribuetur

$$\frac{n\,dz}{4cc\,.\,\overline{nn-m}\,.\,z^{\frac{4-s}{2}}\,.\,\overline{gz+\frac{k}{2c}}^{\,\frac{s}{2}}\,.\,\sqrt{m+2nz+zz}}$$

$$\frac{+\,dz}{4gc\,.\,\overline{nn-m}\,.\,z^{\frac{2-s}{2}}\,.\,\overline{gz+\frac{k}{2c}}^{\,\frac{s}{2}}\,.\,\sqrt{m+2mz+zz}}$$

$$\frac{-\,dz}{4cc\,.\,\overline{nn-m}\,.\,z^{\frac{4-s}{2}}\,.\,\overline{gz+\frac{k}{2c}}^{\,\frac{s}{2}}}\,.$$ Tertia ex his formulis integratur sine auxilio sectionum conicarum. Prima, & secunda ex N. XXVII per arcus ellypticos, & hyperbolicos integrationem accipiunt.

Idem semper invenies, si t sit numerus quilibet integer & positivus. Etenim divisa formula in plures, aliæ per solas quadraturas circuli, & hyperbolæ construentur, alias N. XXVII. complectetur.

XXXVIII. Quæ hactenus explicata sunt, ostendunt, quando nam construi possit per arcus sectionum conicarum formula

$$\frac{dx}{\overline{a+bx+cx^2}^{\,\frac{r}{2}}\,.\,\overline{f+gx+hx^2}^{\,\frac{s}{2}}}\,.$$ Etenim hæc ita potest

ex-

exponi $\dfrac{dx}{\overline{a+bx+cxx}^{\frac{r+s}{2}} \cdot \overline{\dfrac{b}{c}+\dfrac{g-\frac{bb}{c}.x+f-\frac{ab}{c}}{a+bx+cx^2}}^{\frac{s}{2}}}$.

Quum supponamus r, s esse numeros impares, manifestum est, fore $\frac{r+s}{2}$ numerum integrum, qui fieri potest $=t$. Supra demonstravimus formulam semper esse in potestate, si $t=1$. Hoc autem contingit primo si $s=r=1$; secundo si r sit positiva, s negativa ita ut $r+s=2$, ut si $r=7$, $s=-5$; tertio si r negativa sit, s positiva, ita ut $s+r=2$, ut si $s=5$, $r=-3$. In his omnibus casibus, formula construitur per rectificationem sectionum conicarum.

XXXIX. Præterea ostendimus ignotum esse, quo pacto formula construatur, si aut $t=0$, aut t sit negativa. Hoc autem accidit, si ex duabus speciebus r, s una negativa sit, altera positiva, & vel sint æquales, vel negativa superet positivam, aut utraque sit negativa.

XL. Demum demonstravimus, arcus sectionum conicarum semper præbere integrationem formulæ, quotiescumque t sit positiva. Quod obtinebis, si aut utraque s, r positiva sit, aut existente una positiva, & altera negativa, positiva excedat negativam.

XLI. Si in formula $x^t dx \cdot \overline{a+bx+cxx}^{\frac{r}{3}}$, t sit numerus integer, & positivus, r vero vel positivus, vel negativus, quem tamen non metiatur ternarius, arcus sectionum conicarum ejus integrationi sufficient. Usurpanda est substitutio $a+bx+cxx=cz^3$: ergo $x^2+\frac{b}{c}x=z^3-\frac{a}{c}$, sive

$$x=\frac{-b}{2c}+\sqrt{z^3-\frac{a}{c}+\frac{bb}{4cc}}, \;\&\; dx=\frac{3z^2dz}{2\sqrt{z^3-\frac{a}{c}+\frac{bb}{4cc}}}.$$

Qua-

Quapropter factis substitutionibus formula in hanc mutatur

$$\frac{\overline{\frac{-b}{2c}+\sqrt{z^3-\frac{a}{c}+\frac{bb}{4cc}}}^{\,t}\cdot 3c^{\frac{r}{3}}z^{r+2}dz}{2\sqrt{z^3-\frac{a}{c}+\frac{bb}{4cc}}}.$$

Quum t sit numerus integer, & positivus, elevato binomio ad potestatem t, factaque multiplicatione, plures formulæ exorientur, quarum aliquæ erunt integrabiles, aliæ semper construentur per arcus ellypticos, & hyperbolicos ex N. XXVI.

XLII. Superior formula, posita $b=o$, reduceretur ad arcus sectionum conicarum, tametsi t foret numerus integer, & negativus ex N. XXVI, quotiescumque per notas quadraturas non haberetur.

XLIII. Progredior ad formulam $x^t dx.\overline{a+bx+cx^2}^{\frac{r}{4}}$, in qua t est numerus integer, & positivus, r vero impar vel positivus vel negativus. Pono $a+bx+cx^2=cz^2$: ergo $x^2+\frac{b}{c}x=zz-\frac{a}{c}$, sive $x=-\frac{b}{2c}+\sqrt{zz-\frac{a}{c}+\frac{bb}{4cc}}$, & $dx=\frac{zdz}{\sqrt{zz-\frac{a}{c}+\frac{bb}{4cc}}}$. Quare formula in hanc mutabitur

$$\overline{\frac{-b}{2c}+\sqrt{zz-\frac{a}{c}+\frac{bb}{4cc}}}^{\,t}\cdot\frac{c^{\frac{r}{4}}z^{\frac{r+2}{2}}dz}{\sqrt{zz-\frac{a}{c}+\frac{bb}{4cc}}},$$

quæ elevato binomio ad potestatem integram t, in plures formulas convertitur: atque hæ omnes, vel sunt integrabiles, vel reducuntur ad N. XIX.

XLIV. Posito $b=o$, etiamsi t sit negativus, formula, nisi integretur aut absolute, aut per notas quadraturas, in summam col-

colligitur per arcus ellypticos, & hyperbolicos ex N. XIX.

XLV. Reliquum est, ut verba faciam de formula

$x^t dx . \overline{a+bx+cxx}^{\frac{r}{6}}$, in qua t est numerus integer, & positivus, r vel positivus vel negativus, qui tamen neque sit par, neque divisibilis per 3, secus enim formula in numeris superioribus contineretur. Usurpetur hæc substitutio

$a+bx+cx^2 = cz^3$, ex qua fit $x^2 + \frac{b}{c}x = z^3 - \frac{a}{c}$: ergo

$x = \frac{-b}{2c} + \sqrt{z^3 - \frac{a}{c} + \frac{bb}{4cc}}$, & sumptis differentiis

$dx = \frac{3z^2 dz}{2\sqrt{z^3 - \frac{a}{c} + \frac{bb}{4cc}}}$. Itaque formula in hanc mutabitur

$\overline{\frac{-b}{2c} + \sqrt{z^3 - \frac{a}{c} + \frac{bb}{4cc}}}^{t} . \frac{3c^{\frac{r}{6}} z^{2+\frac{r}{2}} dz}{2\sqrt{z^3 - \frac{a}{c} + \frac{bb}{4cc}}}$. Si binomium

elevetur ad potestatem integram t, & fiat multiplicatio, plures formulæ orientur, quarum aliquæ absolute integrabiles erunt, nempe secunda, quarta, sexta, aliæque, quæ tenent sedes pares; aliæ nimirum prima, tertia, quinta, & reliquæ positæ in sedibus imparibus construentur ex N. XXIX.

XLVI. Si in præmissa formula, esset $b = o$, facta substitutione oriretur formula $\frac{3c^{\frac{r}{6}}}{2} z^{2+\frac{r}{2}} dz . \overline{z^3 - \frac{a}{c}}^{\frac{2t-1}{2}}$, quæ est in nostra potestate, licet t sit numerus negativus. Namque si t est impar aut integrabilis erit algebrice, aut per notas quadraturas circuli, & hyperbolæ; si vero t sit par, continetur in eodem N. XXIX.

EPI-

EPISTOLÆ QUATUOR

In quibus aliquot formulæ ad constructionem perducuntur.

VINCENTIUS RICCATUS

PIO FANTONO

Sancti Petronii Canonico

S. P. D.

QUæsisti ex me, Vir Doctissime, ut ex mea methodo deducerem rectificationem Lemniscatæ per arcus ellypticos, & hyperbolicos, quo comparari possit cum illis, quæ jamdiu traditæ sunt a Comite de Fagnanis. Morem tibi geram, ut par est. Sed nisi tibi molestum est, plura addam de integratione formularum, quæ similitudinem non exiguam habent cum illa, per quam lemniscata rectificatur. Ab hac curva exordium ducam.

Sit Lemniscata CLB, cujus (*Fig.* 8.) corda $CL = z$: constat, vocata $CB = a$, arcum directum

$$CL = S\frac{aadz}{\sqrt{aa+zz}.\sqrt{aa-zz}},$$ arcum vero inversum

$$BL = S\frac{-aadz}{\sqrt{aa+zz}.\sqrt{aa-zz}}.$$ Formulæ istæ integrantur ex N. XII primæ disquisitionis: atque hæc oritur constructio. Perpendicularis CB ducatur $CA = a\sqrt{2}$, & positis semiaxibus CA, CB describatur ellypsis ADB: tum posito semiaxe CB delineetur hyperbola æquilatera BVO. Abscinde $CF = \sqrt{aa-zz}.\sqrt{2}$, cui respondet ordinata $FD = CG = z$, & determina arcum BD. Deinde abscinde $CS = \frac{a\sqrt{aa+zz}}{z\sqrt{2}}$, & determina arcum BO.

Ha-

Habebimus $BL = S \frac{-aadz}{\sqrt{aa+zz} \cdot \sqrt{aa-zz}} = H - \frac{\sqrt{aa+zz} \cdot \sqrt{aa-zz}}{z} + BD + BO$. Ad determinandam quantitatem additam H, adverte, omnia evanescere facta $z = a$. Ergo $H = o$. Igitur

$$BL = S \frac{-aadz}{\sqrt{aa+zz} \cdot \sqrt{aa-zz}} = \frac{-\sqrt{aa+zz} \cdot \sqrt{aa-zz}}{z} +$$

$BD + BO$: quæ formula coincidit cum illa, quæ tradita est secundo loco a Comite de Fagnanis. Nam quum recta $CS = \frac{a\sqrt{aa+zz}}{z\sqrt{2}}$, si a centro C ad punctum O intelligatur ducta recta CO, hæc invenietur $= \frac{aa}{z}$, prout ab illo Auctore determinatur.

Rectificatio hæc lemniscatæ hoc habet incommodi, quod, si punctum L valde accedat ad punctum C, vel maxime augetur tum arcus BO, tum quantitas algebraica $\frac{\sqrt{aa+zz} \cdot \sqrt{aa-zz}}{z}$; imo coincidente puncto L cum C utraque evadit infinita. Remedium hisce incommodis afferemus, si advertamus, quæ demonstrata sunt tum in litteris ad Mariscottum, tum in prima disquisitione: nimirum determinandi sunt arcus VO, VN, quorum differentia sit rectificabilis, & pro arcu BO substituendus est in formula arcus BN. Hanc ob rem abscinde primum $CT = a\sqrt{1 + \frac{1}{\sqrt{2}}}$, & determina arcum constantem BV; tum seca $CP = \frac{aa}{\sqrt{aa-zz}}$, & determina arcum BN. His positis erit $\frac{aa\sqrt{aa+zz}}{z\sqrt{aa-zz}} - a \cdot \sqrt{\sqrt{2}+1} = VO - VN$, sive

$= BO + BN - 2BV$. Itaque opportune ſubſtituto in ſuperiore æquatione valore arcus BO prodibit

$$BL = S\frac{-aadz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{-\sqrt{aa+zz}.\sqrt{aa-zz}}{z}$$

$$+ \frac{aa\sqrt{aa+zz}}{z\sqrt{aa-zz}} - a.\overline{\sqrt{2}+1} + BD - BN + 2BV\text{, ſive}$$

$$BL = S\frac{-aadz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{z\sqrt{aa+zz}}{\sqrt{aa-zz}} - a.\overline{\sqrt{2}+1}$$

$+ BD - BN + 2BV$.

Fiat nunc $z = 0$, & obtinebimus quadrantem lemniſcatæ

$BLC = -a.\overline{\sqrt{2}+1} + BDA + 2BV$.

Dematur ex hac æquatione æquatio ſuperior, & orietur

$$CL = S\frac{aadz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{-z\sqrt{aa+zz}}{\sqrt{aa-zz}} + AD + BN,$$

quæ coincidit cum prima rectificatione Comitis de Fagnanis. Nam poſita $CP = \frac{aa}{\sqrt{aa-zz}}$, erit $CN = \frac{a\sqrt{aa+zz}}{\sqrt{aa-zz}}$, prout determinatur ab Auctore.

In hac ſi ſubſtituas valorem arcus BN datum per BO, obtinebis

$$CL = S\frac{aadz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{-z\sqrt{aa+zz}}{\sqrt{aa-zz}} + \frac{aa\sqrt{aa+zz}}{z\sqrt{aa-zz}}$$

$$- a.\overline{\sqrt{2}+1} + AD - BO + 2BV\text{, ſive}$$

$$CL = S\frac{aadz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{\sqrt{aa+zz}.\sqrt{aa-zz}}{z}$$

$$- a.\overline{\sqrt{2}+1} + AD - BO + 2BV.$$

Si fiat $z = a$, orietur quadrans Lemniſcatæ

$CLB = -a.\overline{\sqrt{2}+1} + ADB + 2BV$, prorſus ut antea.

For-

Formulæ quatuor, quas pro rectificanda lemniscata invenimus, in alias transmutari facile possunt, si advertas, assignari posse arcum ellypticum AE ita, ut differentia arcuum BD, AE sit integrabilis. Assignatur autem hoc modo. Quando $CG = z$, abscindatur $CH = \frac{a\sqrt{aa-zz}}{\sqrt{aa+zz}}$, ex qua proveniet $CI = \frac{2az}{\sqrt{aa+zz}}$, & determinetur arcus AE. Habebimus $\frac{z\sqrt{aa-zz}}{\sqrt{aa+zz}} = BD - AE$. Quapropter formulæ inventæ in has mutabuntur

$$BL = S\frac{-aadz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{-aa\sqrt{aa-zz}}{z\sqrt{aa+zz}} + AE + BO,$$

$$BL = S\frac{-aadz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{2aaz}{\sqrt{aa+zz}.\sqrt{aa-zz}}$$

$-a.\sqrt{2}+1 + AE - BN + 2BV$. Ad transformandas alias duas, vocato quadrante ellyptico $ADB = Q$, adverte, fore $BD = Q - AD$, $AE = Q - BE$: ergo $BD - AE = BE - AD$: atqui $\frac{z\sqrt{aa-zz}}{\sqrt{aa+zz}} = BD - AE$: ergo $\frac{z\sqrt{aa-zz}}{\sqrt{aa+zz}} = BE - AD$. Quare provenient formulæ

$$CL = S\frac{aadz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{-2aaz}{\sqrt{aa+zz}.\sqrt{aa-zz}}$$

$+ BE + BN$

$$CL = S\frac{aadz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{aa\sqrt{aa-zz}}{z\sqrt{aa+zz}} - a.\sqrt{2}+1$$

$+ BE - BO + 2BV$.

Transeo ad formulam $\frac{zzdz}{\sqrt{aa+zz}.\sqrt{aa-zz}}$, quæ, ut constat ex N. III disquisitionis secundæ in duas dividitur hoc modo

$$\frac{zz\,dz}{\sqrt{aa+zz}\cdot\sqrt{aa-zz}}=\frac{dz\sqrt{aa+zz}}{\sqrt{aa-zz}}-\frac{aa\,dz}{\sqrt{aa+zz}\cdot\sqrt{aa-zz}}.$$

Harum ultima paullo ante integrata est; est enim illa ipsa, per quam integratur lemniscata. Prima integratur per rectificationem ellypsis, ut traditum est N. II primæ disquisitionis. Constructio facillima est. Nam in eadem ellypsi, cujus semiaxis major $CA=a\sqrt{2}$, minor $CB=a$, in minore abscissa $CG=z$, erit $S\,\frac{dz\sqrt{aa+zz}}{\sqrt{aa-zz}}=AD$, & $S\,\frac{-dz\sqrt{aa+zz}}{\sqrt{aa-zz}}=BD$.

His positis jam habes constructionem formulæ $\frac{zz\,dz}{\sqrt{aa+zz}\cdot\sqrt{aa-zz}}$, quæ, quum arcus elliptici ex contrarietate signorum elidantur, dependet unice ab hyperbolæ rectificatione. Itaque si summatoriam accipias ita, ut facta $z=0$, evanescat, obtinebis

$$S\,\frac{-zz\,dz}{\sqrt{aa+zz}\cdot\sqrt{aa-zz}}=\frac{-z\sqrt{aa+zz}}{\sqrt{aa-zz}}+BN,\ \text{five}$$

$$S\,\frac{-zz\,dz}{\sqrt{aa+zz}\cdot\sqrt{aa-zz}}=\frac{\sqrt{aa+zz}\cdot\sqrt{aa-zz}}{z}-a\cdot\sqrt{2+1}$$

$-BO+2BV$. In hac si ponas $z=a$, invenies

$$S\,\frac{-zz\,dz}{\sqrt{aa+zz}\cdot\sqrt{aa-zz}}=-a\cdot\sqrt{2}+1+2BV.$$

Simili modo si velis omnia nihilo æqualia fieri, posita $z=a$, nancisceris

$$S\,\frac{zz\,dz}{\sqrt{aa+zz}\cdot\sqrt{aa-zz}}=\frac{-\sqrt{aa+zz}\cdot\sqrt{aa-zz}}{z}+BO,\ \text{seu}$$

$$S\,\frac{zz\,dz}{\sqrt{aa+zz}\cdot\sqrt{aa-zz}}=\frac{z\sqrt{aa+zz}}{\sqrt{aa-zz}}-a\cdot\sqrt{2+1}-BN+$$

$2BV$, in qua si facias $z=0$, oritur

$$S\,\frac{zz\,dz}{\sqrt{aa+zz}\cdot\sqrt{aa-zz}}=-a\cdot\sqrt{2}+1+2BV\ \text{prorsus ut antea.}$$

Nunc

Nunc ostendendum assumo, quomodo per arcus conicarum sectionum integrationem accipiat formula generalis

$$\frac{z^t dz}{a^{t-2}\sqrt{aa+zz}\cdot\sqrt{aa-zz}},$$ existente t numero pari positivo, vel negativo. Revocans methodum, quam adhibui N. VI disquisitionis secundæ, accipio differentiam formulæ

$$\frac{z^{r+1}\sqrt{aa+zz}\cdot\sqrt{aa-zz}}{a^{r+2}}$$ in hunc modum

$$D\frac{z^{r+1}\sqrt{aa+zz}\cdot\sqrt{aa-zz}}{a^{r+2}} = \frac{\overline{r+1}\cdot z^r dz\sqrt{aa+zz}\cdot\sqrt{aa-zz}}{a^{r+2}}$$

$$+\frac{z^{r+2}dz\sqrt{aa-zz}}{a^{r+2}\sqrt{aa+zz}} - \frac{z^{r+2}dz\sqrt{aa+zz}}{a^{r+2}\sqrt{aa-zz}}$$: atque tribus formulis ad eamdem denominationem redactis invenio

$$D\frac{z^{r+1}\sqrt{aa+zz}\cdot\sqrt{aa-zz}}{a^{r+2}} = \frac{\overline{r+1}\cdot a^4 z^r dz - \overline{r-3}\, z^{r+4}dz}{a^{r+2}\sqrt{aa+zz}\cdot\sqrt{aa-zz}}.$$

Vocatis $\sqrt{aa+zz}=M$, $\sqrt{aa-zz}=N$, æquationem ita dispono

$$\frac{z^{r+4}dz}{a^{r+2}MN} = -D\frac{z^{r+1}MN}{\overline{r+3}\cdot a^{r+2}} + \frac{\overline{r+1}\, z^r dz}{\overline{r+3}\cdot a^{r-2}MN}.$$ Nunc gradatim, & successive ponamus r æqualem numeris paribus positivis, & hasce æquationes nanciscemur

$r=0$, $$\frac{z^4 dz}{aaMN} = -D\frac{zMN}{3aa} + \frac{aadz}{3MN}$$

$r=2$, $$\frac{z^6 dz}{a^4 MN} = -D\frac{z^3 MN}{5a^4} + \frac{3z^2 dz}{5MN}$$

$r=4$,

$r = 4$, $\frac{z^8 dz}{a^6 MN} = -D \frac{z^5 MN}{7a^6} + \frac{5z^4 dz}{7aaMN}$

$r = 6$, $\frac{z^{10} dz}{a^8 MN} = -D \frac{z^7 MN}{9a^8} + \frac{7z^6 dz}{9a^4 MN}$, atque ita deinceps progressu satis manifesto.

Quoniam $\frac{z^4 dz}{aaMN}$ datur per $\frac{aadz}{MN}$, & $\frac{z^8 dz}{a^6 MN}$ datur per $\frac{z^4 dz}{aaMN}$, atque ita deinceps, constat, formulam $\frac{z^t dz}{a^{t-2} MN}$, existente t positivo, & pariter pari, integrari integrata formula $\frac{aadz}{MN}$, quæ requirit tum rectificationem ellypsis, tum rectificationem hyperbolæ. Contra $\frac{z^6 dz}{a^4 MN}$ datur per $\frac{zzdz}{MN}$, & $\frac{z^{10} dz}{a^8 MN}$ per $\frac{z^6 dz}{a^4 MN}$: liquet formulam $\frac{z^t dz}{a^{t-2} MN}$, existente t numero positivo, & impariter pari, dependere ab integratione formulæ $\frac{zzdz}{MN}$, quæ poscit solam rectificationem hyperbolæ.

Verum factis opportune substitutionibus inveniemus formulas datas unice per $\frac{aadz}{MN}$, $\frac{zzdz}{MN}$, nempe

$$\frac{z^4 dz}{aaMN} = -D \frac{zMN}{3aa} + \frac{aadz}{3MN}$$

$$\frac{z^6 dz}{a^4 MN} = -D \frac{z^3 MN}{5a^4} + \frac{3z^2 dz}{5MN}$$

$z^8 dz$

$$\frac{z^8 dz}{a^6 MN} = -D\,\overline{\frac{z^5}{7a^5} + \frac{5z}{3.7.a}}\,.\,\frac{MN}{a} + \frac{5aadz}{3.7.MN}$$

$$\frac{z^{10} dz}{a^8 MN} = -D\,\overline{\frac{z^7}{9a^7} + \frac{7z^3}{5.9.a^3}}\,.\,\frac{MN}{a} + \frac{3.7z^2 dz}{5.9MN}$$

$$\frac{z^{12} dz}{a^{10} MN} = -D\,\overline{\frac{z^9}{11a^9} + \frac{9z^5}{7.11.a^5} + \frac{5.9z}{3.7.11.a}}\,.\,\frac{MN}{a} + \frac{5.9a^2 dz}{3.7.11MN}$$

$$\frac{z^{14} dz}{a^{12} MN} = -D\,\overline{\frac{z^{11}}{13a^{11}} + \frac{11z^7}{9.13.a^7} + \frac{7.11.z^3}{5.9.13.a^3}}\,.\,\frac{MN}{a} + \frac{3.7.11z^2 dz}{5.9.13.MN},$$

quarum formularum progressus perspicuus est.

Ex his integrationem formulæ generalis eliciemus. Nam si t sit numerus pariter par, hæc exorietur formula

$$\frac{z^t dz}{a^{t-2} MN} = -D\,\overline{\frac{z^{t-3}}{\overline{t-1}.a^{t-3}} + \frac{\overline{t-3}.z^{t-7}}{\overline{t-1}.\overline{t-5}.a^{t-7}} + \frac{\overline{t-3}.\overline{t-7}.z^{t-11}}{\overline{t-1}.\overline{t-5}.\overline{t-9}.a^{t-11}} \ldots}$$

$$\overline{\ldots\frac{\overline{t-3}.\overline{t-7}\ldots 5.z}{\overline{t-1}.\overline{t-5}\ldots 3.a}}\,.\,\frac{MN}{a} + \frac{\overline{t-3}.\overline{t-7}\ldots 5aadz}{\overline{t-1}.\overline{t-5}\ldots 3MN}.$$

Si vero t fuerit numerus impariter par, hæc formula exorietur

$$\frac{z^t dz}{a^{t-2} MN} = -D\,\overline{\frac{z^{t-3}}{\overline{t-1}.a^{t-3}} + \frac{\overline{t-3}.z^{t-7}}{\overline{t-1}.\overline{t-5}.a^{t-7}} + \frac{\overline{t-3}.\overline{t-7}.z^{t-11}}{\overline{t-1}.\overline{t-5}.\overline{t-9}.a^{t-11}} \ldots}$$

$$\overline{\ldots\frac{\overline{t-3}.\overline{t-7}\ldots 7.z^3}{\overline{t-1}.\overline{t-5}\ldots 5.a^3}}\,.\,\frac{MN}{a} + \frac{\overline{t-3}.\overline{t-7}\ldots 3.z^2 dz}{\overline{t-1}.\overline{t-5}\ldots 5MN}.$$

Ut eamdem formulam integremus, quum t est numerus quidem par, sed negativus, æquationem supra inventam hac alia ratione distribuo

$z^r dz$

$$\frac{z^r dz}{a^{r-2} MN} = D \frac{z^{r+1} . MN}{r+1 . a^{r+2}} + \frac{r+3 . z^{r+4} dz}{r+1 . a^{r+2} MN}$$. Nunc gradatim, & successive ponamus r æqualem numeris paribus, & negativis, atque has æquationes inveniemus

$$r = -2, \frac{a^4 dz}{zz MN} = -D \frac{MN}{z} - \frac{zz dz}{MN}$$

$$r = -4, \frac{a^6 dz}{z^4 MN} = -D \frac{a^2 MN}{3z^3} + \frac{aa dz}{3 MN}$$

$$r = -6, \frac{a^8 dz}{z^6 MN} = -D \frac{a^4 MN}{5z^5} + \frac{3a^4 dz}{5z^2 MN}$$

$$r = -8, \frac{a^{10} dz}{z^8 MN} = -D \frac{a^6 MN}{7z^7} + \frac{5a^6 dz}{7z^4 MN}$$, atque ita deinceps progressu satis manifesto.

Quoniam a formula $\frac{zz dz}{MN}$ dependet $\frac{a^4 dz}{zz MN}$, a qua dependet $\frac{a^8 dz}{z^6 MN}$, atque ab hac $\frac{a^{12} dz}{z^{10} MN}$, constat formulam $\frac{a^{t+2} dz}{z^t MN}$, si t sit numerus impariter par, dependere a sola hyperbolæ rectificatione. Simili ratione probabis, eamdem formulam $\frac{a^{t+2} dz}{z^t MN}$, existente t nnmero pariter pari, tamdem dependere a formula $\frac{aa dz}{MN}$, atque adeo postulare tum hyperbolæ, tum ellypseos rectificationem.

Si

Si opportune fiant substitutiones, habebuntur

$$\frac{a^4 dz}{zzMN} = -D\frac{MN}{z} - \frac{zzdz}{MN}$$

$$\frac{a^6 dz}{z^4 MN} = -D\frac{aaMN}{3z^3} + \frac{aadz}{3MN}$$

$$\frac{a^8 dz}{z^6 MN} = -D\overline{\frac{a^5}{5z^5} + \frac{3a}{5z}}.\frac{MN}{a} - \frac{3z^2 dz}{5MN}$$

$$\frac{a^{10} dz}{z^8 MN} = -D\overline{\frac{a^7}{7z^7} + \frac{5a^3}{7.3.z^3}}.\frac{MN}{a} + \frac{5a^2 dz}{7.3.MN}$$

$$\frac{a^{12} dz}{z^{10} MN} = -D\overline{\frac{a^9}{9z^9} + \frac{7a^5}{9.5.z^5} + \frac{7.3.a}{9.5z}}.\frac{MN}{a} - \frac{7.3.z^2 dz}{9.5.MN}$$

$$\frac{a^{14} dz}{z^{12} MN} = -D\frac{a^{11}}{11.z^{11}} + \frac{9a^7}{11.7.z^7} + \frac{9.5a^3}{11.7.3.z^3}.\frac{MN}{a} + \frac{9.5a^2 dz}{11.7.3.MN},$$

ex quo progressu aliæ formulæ facillime inveniri possunt.

Imo ex his generalis formulæ $\frac{a^{t+2} dz}{z^t MN}$ deducimus integrationem, dummodo t sit numerus par. Etenim si t sit impariter par, hæc orietur formula

$$\frac{a^{t+2} dz}{z^t MN} = -D\overline{\frac{a^{t-1}}{\overline{t-1}.z^{t-1}} + \frac{\overline{t-3}.a^{t-5}}{\overline{t-1}.\overline{t-5}.z^{t-5}} + \frac{\overline{t-3}.\overline{t-7}.a^{t-9}}{\overline{t-1}.\overline{t-5}.\overline{t-9}z^{t-9}} \ldots\ldots}$$

$$\ldots\ldots\frac{\overline{t-3}.\overline{t-7}\ldots\ldots 3a}{\overline{t-1}.\overline{t-5}\ldots\ldots 5z}.\frac{MN}{a} - \frac{\overline{t-3}.\overline{t-7}\ldots\ldots 3z^2 dz}{\overline{t-1}.\overline{t-5}\ldots\ldots 5MN}.$$

Si vero t fuerit numerus pariter par, formula hæc exurget

$$\frac{a^{t+2}dz}{z^{t}MN} = -D\frac{a^{t-1}}{\overline{t-1}.z^{t-1}} + \frac{\overline{t-3}.a^{t-5}}{\overline{t-1}.\overline{t-5}.z^{t-5}} + \frac{\overline{t-3}.\overline{t-7}.a^{t-9}}{\overline{t-1}.\overline{t-5}.\overline{t-9}.z^{t-9}} \dots$$

$$\dots\frac{\overline{t-3}.\overline{t-7}\dots 5.a^{3}}{\overline{t-1}.\overline{t-5}\dots 3.z^{3}}.\frac{MN}{a} + \frac{\overline{t-3}.\overline{t-7}\dots 5\,aadz}{\overline{t-1}.\overline{t-5}\dots 3\,MN}.$$

Hæc satis superque docent, quo pacto integretur formula $\frac{z^{t}dz}{a^{t-2}MN}$, si t sit numerus par vel positivus, vel negativus: quæ formula solum realis erit, si z consistat intra limites $z = \pm a$, $z = \pm o$. Hac vero formula integrata per arcus sectionum conicarum, facili negotio integrabis formulas aliquot, quæ per alias methodos ægre admodum evolventur. Uno, aut altero exemplo rem tibi aperiam.

Sit proposita ad integrandum formula $\frac{\overline{5a^{4}-7z^{4}}.z^{4}dz}{a^{6}MN}$, in qua, ut supra, $M = \sqrt{aa+zz}$, $N = \sqrt{aa-zz}$. Hanc divide in duas hoc modo $\frac{5z^{4}dz}{aaMN} - \frac{7z^{8}dz}{a^{6}MN}$: quæ ex supra dictis hujusmodi obtinent integrationem

$$5\,S\frac{z^{4}dz}{aaMN} = \frac{-5zMN}{3aa} + \frac{5}{3}S\frac{aadz}{MN}$$

$$7S\frac{z^{8}dz}{a^{6}MN} = \frac{-z^{5}}{a^{5}} - \frac{5z}{3a}.\frac{MN}{a} + \frac{5}{3}S\frac{aadz}{MN}:$$ igitur facta detractione habebimus $S\frac{\overline{5a^{4}-7z^{4}}.z^{4}dz}{a^{6}MN} = \frac{z^{5}}{a^{5}}.\frac{MN}{a}$, in

qua

qua quantitates transcendentes evanuerunt, & solum relictæ sunt quantitates algebraicæ.

Alterum exemplum præbeat formula

$\frac{7z^8 - 14a^4z^4 + 3a^8 . a^4 dz}{7z^{10}MN}$. Hæc resolvatur in tres, nempe

$\frac{a^4 dz}{z^2 MN} - \frac{2a^8 dz}{z^6 MN} + \frac{3a^{12} dz}{7z^{10} MN}$, quibus hæc convenit integratio

$$S\frac{a^4 dz}{zzMN} = -\frac{MN}{z} - S\frac{zzdz}{MN}$$

$$2S\frac{a^8 dz}{z^6 MN} = \frac{-2a^5}{5z^5} - \frac{6a}{5z} . \frac{MN}{a} - \frac{6}{5} S\frac{zzdz}{MN}$$

$$\frac{3}{7}S\frac{a^{12} dz}{z^{10} MN} = \frac{-a^9}{21z^9} \frac{-a^5}{15z^5} \frac{-a}{5z} . \frac{MN}{a} - \frac{1}{5} S\frac{zzdz}{MN}.$$

Facta demum opportuna additione, & deductione proveniet

$S\frac{7z^8 - 14a^4z^4 + 3a^8 . a^4 dz}{7z^{10}MN} = \frac{-a^9}{21z^9} + \frac{a^5}{3z^5} . \frac{MN}{a}$; evanescunt enim quantitates transcendentes, & formula evadit algebraice integrabilis.

Similiter si advertas $S\frac{aadz}{MN} + S\frac{zzdz}{MN} = S\frac{Mdz}{N}$ æqualem esse arcui ellyptico dumtaxat, nempe arcui AD, sumpta in axe minore abscissa $CG = z$, integrabis aliquot formulas supposita sola ellypseos rectificatione. Duobus exemplis rem declarabo. Assumo integrandam formulam $\frac{77a^{10} + 65z^{10} . z^4 dz}{a^{12}MN}$,

quam

quam in duas divido hoc modo $\frac{77z^4dz}{aaMN} + \frac{65z^{14}dz}{a^{12}MN}$. Integratio primæ hæc erit

$$77\,S\frac{z^4dz}{a^2MN} = \frac{-77z}{3a}\cdot\frac{MN}{a} + \frac{77}{3}S\frac{a^2dz}{MN};$$ alterius integratio supputationibus effectis erit hujusmodi

$$65\,S\frac{z^{14}dz}{a^{12}MN} = \frac{-5z^{11}}{a^{11}} - \frac{55z^7}{9a^7} - \frac{77z^3}{9a^3}\cdot\frac{MN}{a} + \frac{77}{3}S\frac{zzdz}{MN}:$$

ergo formulis simul additis, & substituto arcu AD pro $S\frac{aadz}{MN} + S\frac{zzdz}{MN}$, fiet

$$S\frac{77a^{10}+65z^{10}\cdot z^4dz}{a^{12}MN} = \frac{-5z^{11}}{a^{11}} - \frac{55z^7}{9a^7} - \frac{77z^3}{9a^3} - \frac{77z}{3a}\cdot\frac{MN}{a} + \frac{77}{3}AD.$$

Secundum exemplum sufficiet formula

$$\frac{5z^4 - 24aazz + 5a^4\cdot a^4dz}{z^6MN},$$ quæ dividenda est in tres, nempe

$$\frac{5a^4dz}{zzMN} - \frac{24a^6dz}{z^4MN} + \frac{5a^8dz}{z^6MN}.$$ Harum summas habeto

$$5\,S\frac{a^4dz}{zzMN} = \frac{-5MN}{z} - 5\,S\frac{zzdz}{MN}$$

$$24\,S\frac{a^6dz}{z^4MN} = \frac{-8a^2MN}{z^3} + 8\,S\frac{aadz}{MN}$$

$5 S \frac{a^8 dz}{z^6 MN} = \overline{\frac{-a^5}{z^5} - \frac{3a}{z} \cdot \frac{MN}{a}} - 3 S \frac{zz\,dz}{MN}$. Quare si extremæ æquationes in summam colligantur, & dematur media, proveniet

$$S \frac{\overline{5z^4 - 24a^2z^2 + 5a^4} \cdot a^4 dz}{z^6 MN} = \overline{\frac{-a^5}{z^5} + \frac{8a^3}{z^3} - \frac{8a}{z} \cdot \frac{MN}{a}}$$

$$- 8 \,.\, S \frac{zz\,dz}{MN} + S \frac{a^2 dz}{MN},$$ in qua si pro

$S \frac{zz\,dz}{MN} + S \frac{aa\,dz}{MN}$ colloces arcum AD, habebis formulam integratam supposita solius ellypseos rectificatione.

Hæc satis sint. Tuum erit, Vir Clarissime, judicare, quanti hoc genus integrationis faciendum sit. Vale.

Ex Collegio Sanctæ Luciæ duodecimo Kal. Novembris 1757.

VINCENTIUS RICCATUS
JOANNI FRANCISCO MALFATTO
Philosophiæ Experimentalis Professori

S. P. D.

QUum in litteris ad Fantonum datis per sectionum conicarum arcus integraverim formulam

$\frac{z^t dz}{a^{t-2}\sqrt{a^2+z^2}.\sqrt{a^2-z^2}}$ posito t numero pari vel positivo, vel negativo, numquam cogitassem de integratione formulæ prorsus similis $\frac{z^t dz}{a^{t-2}\sqrt{a^2+z^2}.\sqrt{-a^2+z^2}}$, nisi tu, Vir Clarissime, mihi significasses, pergratum tibi fore, si hanc quoque formulam non plane negligerem, & instituto calculo ad constructionem perducerem. Quando nihil tibi petenti negare possum, rem aggredior non invitus: nam quamquam eamdem methodum adhibere, necesse est; tamen novum hoc exemplum theoriam hanc non mediocriter illustrabit. Hoc primum adverte, formulam, de qua agere incipio, realem esse non posse, nisi z consistat intra limites $z = \pm a$, $z = \pm \infty$.

Si sequaris methodum, quæ indicatur in disquisitione secunda N. I, invenies

$$\frac{aadz}{\sqrt{aa+zz}.\sqrt{-aa+zz}} = \frac{dz\sqrt{aa+zz}}{2\sqrt{-aa+zz}} - \frac{dz\sqrt{-aa+zz}}{2\sqrt{aa+zz}}.$$

Ex methodo N. III ejusdem disquisitionis invenies

$$\frac{zzdz}{\sqrt{aa+zz}.\sqrt{-aa+zz}} = \frac{dz\sqrt{aa+zz}}{2\sqrt{-aa+zz}} + \frac{dz\sqrt{-aa+zz}}{2\sqrt{aa+zz}}.$$

Formulæ duæ $\frac{dz\sqrt{aa+zz}}{2\sqrt{-aa+zz}}$, $\frac{dz\sqrt{-aa+zz}}{2\sqrt{aa+zz}}$ in prima disquisitiositione integrantur, prima N. XVII per arcus ellypticos, & hyperbolicos, secunda N. VI per solos arcus ellypticos.

Primæ ex duabus formulis constructio hæc nascitur. Descripta ellypsi A D B, (*Fig.* 8.) cujus semiaxis major $CA = a\sqrt{2}$, minor $CB = a$: abscinde in axe minore $CG = \frac{a\sqrt{-aa+zz}}{\sqrt{aa+zz}}$, & ducta ordinata G D, determina arcus A D, B D. Tum descripta hyperbola æquilatera B O, cujus semiaxis $CB = a$, accipe $CS = \frac{az}{\sqrt{-aa+zz}}$, & determina arcum B O. Fiet

$$S\frac{dz\sqrt{aa+zz}}{2\sqrt{-aa+zz}} = \frac{z\sqrt{-aa+zz}}{2\sqrt{aa+zz}} + \frac{2aaz}{\sqrt{aa+zz}\cdot\sqrt{-aa+zz}} - BD - \frac{AD}{2} - BO.$$

Alterius formulæ hæc constructio obtinetur. In eadem ellypsi accipe, ut supra, $CG = \frac{a\sqrt{-aa+zz}}{\sqrt{aa+zz}}$, & habebimus

$$S\frac{dz\sqrt{-aa+zz}}{2\sqrt{aa+zz}} = \frac{z\sqrt{-aa+zz}}{2\sqrt{aa+zz}} - \frac{AD}{2}.$$

Si duarum formularum differentiam sumis, obtines

$$S\frac{aadz}{\sqrt{aa+zz}\cdot\sqrt{-aa+zz}} = \frac{2aaz}{\sqrt{aa+zz}\cdot\sqrt{-aa+zz}} - BD - BO.$$

Si earumdem formularum capias summam, invenies

$$S\frac{zzdz}{\sqrt{aa+zz}\cdot\sqrt{-aa+zz}} = \frac{z\sqrt{-aa+zz}}{\sqrt{aa+zz}} + \frac{2aaz}{\sqrt{aa+zz}\cdot\sqrt{-aa+zz}} - BD - AD - BO.$$

In hac formula, quum $BD + AD$ det integrum quadrantem ellypticum, qui constans est, tuto omitti potest, quia in accipienda summatoria ea quantitas addenda est, quam circumstantiæ requirunt. Itaque habebimus ad unam redactis duabus formulis algegebraicis $S\frac{zz\,dz}{\sqrt{aa+zz}\cdot\sqrt{-aa+zz}} = \frac{z\sqrt{aa+zz}}{\sqrt{-aa+zz}} - BO$.

In ambabus formulis, facta $z = a$, tam quantitas algebraica, quam arcus BO fit infinitus: qua de re accipienda esset differentia inter quantitates infinitas, quæ finita esse potest. Ut difficultas hæc arceatur, per ea, quæ dicta sunt cum in epistola ad Mariscottum, tum in prima disquisitione, determinandi sunt arcus VO, VN, quorum differentia rectificabilis sit, & pro arcu BO substituendus est ejus valor datus per BN. Hanc ob rem sume $CT = a\sqrt{1+\frac{1}{\sqrt{2}}}$; tum abscinde $CP = \sqrt{\frac{aa+zz}{2}}$. Habebis $VO - VN = \frac{z\sqrt{aa+zz}}{\sqrt{-aa+zz}} - a.\overline{\sqrt{2}+1}$. Ergo $BO = \frac{z\sqrt{aa+zz}}{\sqrt{-aa+zz}} - a.\overline{\sqrt{2}+1} + 2BV - BN$. Qui valor substitutus in duabus formulis superioribus exhibet

$$S\frac{aa\,dz}{\sqrt{aa+zz}\cdot\sqrt{-aa+zz}} = \frac{2aaz}{\sqrt{aa+zz}\cdot\sqrt{-aa+zz}} - \frac{z\sqrt{aa+zz}}{\sqrt{aa-zz}} + a.\overline{\sqrt{2}+1} - BD - 2BV + BN,$$ sive

$$S\frac{aa\,dz}{\sqrt{aa+zz}\cdot\sqrt{-aa+zz}} = \frac{-z\sqrt{-aa+zz}}{\sqrt{aa+zz}} - a.\overline{\sqrt{2}+1} - 2BV - BD + BN,$$ &

$$S\frac{zz\,dz}{\sqrt{aa+zz}\cdot\sqrt{-aa+zz}} = a.\overline{\sqrt{2}+1} - 2BV + BN.$$

In his æquationibus per ea, quæ supra dicta sunt, quantitates constantes omitti possunt. Si autem deleantur in ultima, inveniemus arcum $BN = S\frac{zz\,dz}{\sqrt{aa+zz}\cdot\sqrt{-aa+zz}}$.

Quo-

Quoniam abscissa $CP=\frac{\sqrt{aa+zz}}{\sqrt{2}}$, facto calculo orietur corda $CN=z$. Quare expressio tradita illa ipsa est, quam docuit Clarissimus Fagnanus.

Nunc vero ita determinemus summatorias, ut posita $z=a$, prorsus evanescant. In hac hypothesi formulæ erunt

$$S\frac{aadz}{\sqrt{aa+zz}.\sqrt{-aa+zz}}=\frac{-z\sqrt{-aa+zz}}{\sqrt{aa+zz}}-AD+BN$$

$$S\frac{zzdz}{\sqrt{aa+zz}.\sqrt{-aa+zz}}=BN.$$ Si fiat $z=\infty$, secunda ex his summatoriis infinitam æquat quantitatem. Sed prima includit differentiam duarum quantitatum infinitarum, quæ finita esse potest.

Quapropter pro BN substituamus ejus valorem datum per BO, & fiet

$$S\frac{aadz}{\sqrt{aa+zz}.\sqrt{-aa+zz}}=\frac{2aaz}{\sqrt{aa+zz}.\sqrt{-aa+zz}}-a.\sqrt{2}+1+2BV-AD-BO$$

$$S\frac{zzdz}{\sqrt{aa+zz}.\sqrt{-aa+zz}}=\frac{z\sqrt{-aa+zz}}{\sqrt{aa+zz}}-a.\overline{\sqrt{2}+1}$$ $+2BV-BO$. In harum prima, facta $z=\infty$, quantitas algebraica, & BO fit $=o$, arcus AD evadit quadrans ellypsis ADB: ergo habetur

$$S\frac{aadz}{\sqrt{aa+zz}.\sqrt{-aa+zz}}=-a.\overline{\sqrt{2}+1}+2BV-ADB.$$

Secunda autem valorem habet infinitum. Possem formulas, quæ dependent a rectificatione ellypsis in alias convertere, determinatis illis arcubus, quorum differentia integrationem recipit algebraicam. Sed quoniam nihil fiunt elegantiores formulæ, ut brevitati consulam, hasce trasformationes omitto.

Integratis, constructisque duabus formulis revocans methodum usurpatam N. VI disquisitionis secundæ sumo differentiam formulæ

S z^{r+1}

$\frac{z^{r+1}\sqrt{aa+zz}\cdot\sqrt{-aa+zz}}{a^{r+2}}$ hoc modo

$$D\,\frac{z^{r+1}\sqrt{aa+zz}\cdot\sqrt{-aa+zz}}{a^{r+2}} = \frac{\overline{r+1}\cdot z^{r}dz\sqrt{aa+zz}\cdot\sqrt{-aa+zz}}{a^{r+2}}$$

$$+\frac{z^{r+2}dz\sqrt{-aa+zz}}{a^{r+2}\cdot\sqrt{aa+zz}} + \frac{z^{r+2}dz\sqrt{aa+zz}}{a^{r+2}\cdot\sqrt{-aa+zz}}$$: atque tribus hisce formulis ad eamdem denominationem redactis habeo

$$D\,\frac{z^{r+1}\sqrt{aa+zz}\cdot\sqrt{-aa+zz}}{a^{r+2}} =$$

$$\frac{-(\overline{r+1}\cdot a^{4}z^{r}dz+\overline{r+3}\,z^{r+4}dz}{a^{r+2}\cdot\sqrt{aa+zz}\cdot\sqrt{-aa+zz}}.$$

Vocatis $\sqrt{aa+zz}=M$, $\sqrt{-aa+zz}=N$, æquationem ita distribuo

$$\frac{z^{r+4}dz}{a^{r+2}MN} = D\,\frac{z^{r+1}MN}{\overline{r+3}\cdot a^{r+2}} + \frac{\overline{r+1}\cdot z^{r}dz}{\overline{r+3}\cdot a^{r-2}MN}.$$ Nunc pro r colloco successive numeros pares, atque has æquationes nanciscor

$$r=0,\ \frac{z^{4}dz}{aaMN} = D\,\frac{zMN.}{3a^{2}} + \frac{aadz}{3MN}$$

$$r=2,\ \frac{z^{6}dz}{a^{4}MN} = D\,\frac{z^{3}MN}{5a^{4}} + \frac{3z^{2}dz}{5MN}$$

$$r=4,\ \frac{z^{8}dz}{a^{6}MN} = D\,\frac{z^{5}MN}{7a^{6}} + \frac{5z^{4}dz}{7a^{2}MN}$$

$r=6,$

$r=6$, $\frac{z^{10}dz}{a^8 MN} = D\frac{z^7 MN}{9a^8} + \frac{7z^6 dz}{9a^4 MN}$, atque ita deinceps progressu satis manifesto.

Quum $\frac{z^4 dz}{a^2 MN}$ detur per $\frac{aadz}{MN}$, $\frac{z^8 dz}{a^6 MN}$ detur per $\frac{z^4 dz}{aaMN}$, atque ita deinceps; constat $\frac{z^t dz}{a^{t-2} MN}$, existente t numero pariter pari, integrari integrata formula $\frac{aadz}{MN}$, quæ postulat cum ellypsis, tum hyperbolæ rectificationem. Contra $\frac{z^6 dz}{a^4 MN}$ datur per $\frac{zzdz}{MN}$, $\frac{z^{10} dz}{a^8 MN}$ datur per $\frac{z^6 dz}{a^4 MN}$, atque ita deinceps: quare si t sit numerus impariter par, $\frac{z^t dz}{a^{t-2} MN}$ dependet ab integratione formulæ $\frac{zzdz}{MN}$; quæ solius hyperbolæ rectificatione contenta est.

Peractis opportune substitutionibus formulas inquiramus datas unice per $\frac{aadz}{MN}$, $\frac{zzdz}{MN}$. Inveniemus autem

$$\frac{z^4 dz}{a^2 MN} = D\frac{zMN}{3aa} + \frac{aadz}{3MN}$$

$$\frac{z^6 dz}{a^4 MN} = D\frac{z^3 MN}{5a^4} + \frac{3zzdz}{5MN}$$

$$\frac{z^{8}dz}{a^{6}MN} = D\,\overline{\frac{z^{5}}{7a^{5}} + \frac{5z}{3.7.a}}\,.\,\frac{MN}{a} + \frac{5a^{2}dz}{3.7.MN}$$

$$\frac{z^{10}dz}{a^{8}MN} = D\,\overline{\frac{z^{7}}{9a^{7}} + \frac{7z^{3}}{5.9.a^{3}}}\,.\,\frac{MN}{a} + \frac{3.7z^{2}dz}{5.9MN}$$

$$\frac{z^{12}dz}{a^{10}MN} = D\,\overline{\frac{z^{9}}{11.a^{9}} + \frac{9z^{5}}{7.11.a^{5}} + \frac{5.9.z}{3.7.11.a}}\,.\,\frac{MN}{a} + \frac{5.9a^{2}dz}{3.7.11MN}$$

$$\frac{z^{14}dz}{a^{12}MN} = D\,\frac{z^{11}}{13.a^{11}} + \frac{11.z^{7}}{9.13.a^{7}} + \frac{7.11.z^{3}}{5.9.13.a^{3}}\,.\,\frac{MN}{a} + \frac{3.7.11.z^{2}dz}{5.9.13.MN},$$

quarum æquationum progressus cuilibet obvius est.

Ex his formulæ generalis $\frac{z^{t}dz}{a^{t-2}MN}$ integrationem deducimus, dummodo t sit numerus positivus, & par. Nam si t sit numerus impariter par habebimus

$$\frac{z^{t}dz}{a^{t-2}MN} = D\,\overline{\frac{z^{t-3}}{\overline{t-1}.a^{t-3}} + \frac{\overline{t-3}.z^{t-7}}{\overline{t-1}.\overline{t-5}.a^{t-7}} + \frac{\overline{t-3}.\overline{t-7}.z^{t-11}}{\overline{t-1}.\overline{t-5}.\overline{t-9}.a^{t-11}} \ldots}$$

$$\overline{\ldots\ldots \frac{\overline{t-3}.\overline{t-7}\ldots.7.z^{3}}{\overline{t-1}.\overline{t-5}.\overline{t-9}\ldots5.a^{3}}}\,.\,\frac{MN}{a} + \frac{\overline{t-3}.\overline{t-7}\ldots3z^{2}dz}{\overline{t-1}.\overline{t-5}\ldots\ldots5MN}.$$

Si vero fuerit t numerus pariter par, hæc enascetur formula generalis

$$\frac{z^{t}dz}{a^{t-2}MN} = D\,\overline{\frac{z^{t-3}}{\overline{t-1}.a^{t-3}} + \frac{\overline{t-3}.z^{t-7}}{\overline{t-1}.\overline{t-5}.a^{t-7}} + \frac{\overline{t-3}.\overline{t-7}.z^{t-11}}{\overline{t-1}.\overline{t-5}.\overline{t-9}.a^{t-11}} \ldots}$$

$$\overline{\ldots.\frac{\overline{t-3}.\overline{t-7}\ldots\ldots5.z}{\overline{t-1}.\overline{t-5}.\overline{t-9}\ldots.3.a}}\,.\,\frac{MN}{a} + \frac{\overline{t-5}.\overline{t-7}\ldots\ldots5.a^{2}dz}{\overline{t-1}.\overline{t-5}.\overline{t-9}\ldots3MN}.$$

In-

Integranda jam est formula in hypothesi t paris quidem, sed negativi. Hanc ob rem æquationem, quam supra nacti sumus, opus est ita distribuere

$$\frac{z^r dz}{a^{r-2}MN} = -D\frac{z^{r+1}.MN}{r+1.a^{r+2}} + \frac{r+3.z^{r+4}dz}{r+1.a^{r+2}MN},$$ ex qua, posita successive r æquali numeris paribus, & negativis, sequentes orientur æquationes

$$r=-2,\ \frac{a^4 dz}{zzMN} = D\frac{MN}{z} - \frac{zzdz}{MN}$$

$$r=-4,\ \frac{a^6 dz}{z^4MN} = D\frac{a^2MN}{3z^3} + \frac{a^2dz}{3MN}$$

$$r=-6,\ \frac{a^8 dz}{z^6MN} = D\frac{a^4MN}{5z^5} + \frac{3a^4dz}{5zzMN}$$

$$r=-8,\ \frac{a^{10} dz}{z^8MN} = D\frac{a^6MN}{7z^7} + \frac{5a^6dz}{7z^4MN},$$ quarum æquationum progressus patet.

Vides, Vir Clarissime, ex his æquationibus formulam $\frac{a^{t+2}dz}{z^tMN}$, si t sit numerus impariter par, dependere ab integratione formulæ $\frac{zzdz}{MN}$, quæ solius hyperbolæ rectificatione contenta est; contra si t sit numerus pariter par, dependere ab integratione formulæ $\frac{aadz}{MN}$, quæ cum hyperbolæ, tum ellypsis rectificationem requirit.

Ut formulas omnes datas reperiamus per solas $\frac{zzdz}{MN}$, $\frac{a^2dz}{MN}$, opus est opportune substitutionibus uti, per quas hujusmodi æquationes inveniemus.

a^4dz

$$\frac{a^4 dz}{z^2 MN} = D\,\frac{MN}{z} - \frac{zzdz}{MN}$$

$$\frac{a^6 dz}{z^4 MN} = D\,\overline{\frac{a^2 MN}{3z^3}} + \frac{aadz}{3MN}$$

$$\frac{a^8 dz}{z^6 MN} = D\,\overline{\frac{a^5}{5z^5} + \frac{3a}{5z}}\cdot\frac{MN}{a} - \frac{3zzdz}{5MN}$$

$$\frac{a^{10} dz}{z^8 MN} = D\,\overline{\frac{a^7}{7z^7} + \frac{5a^3}{7.3z^3}}\cdot\frac{MN}{a} + \frac{5aadz}{7.3.MN}$$

$$\frac{a^{12} dz}{z^{10} MN} = D\,\overline{\frac{a^9}{9z^9} + \frac{7a^5}{9.5z^5} + \frac{7.3.a}{9.5.z}}\cdot\frac{MN}{a} - \frac{7.3z^2 dz}{9.5MN}$$

$$\frac{a^{14} dz}{z^{12} MN} = D\,\overline{\frac{a^{11}}{11z^{11}} + \frac{9a^7}{11.7.z^7} + \frac{9.5a^3}{11.7.3.z^3}}\cdot\frac{MN}{a} + \frac{9.5aadz}{11.7.3.MN},$$

ex quo progressu aliæ ulteriores formulæ nullo negotio reperientur.

Quod si aveas habere formulas generales, eas ex progressu colliges. Si enim t sit numerus impariter par, habebis

$$\frac{a^{t+2} dz}{z^t MN} = D\,\overline{\frac{a^{t-1}}{\overline{t-1}.z^{t-1}} + \frac{\overline{t-3}.a^{t-5}}{\overline{t-1}.\overline{t-5}.z^{t-5}} + \frac{\overline{t-3}.\overline{t-7}.a^{t-9}}{\overline{t-1}.\overline{t-5}.\overline{t-9}.a^{t-9}}\ldots\ldots}$$

$$\overline{\ldots\ldots\frac{\overline{t-3}.\overline{t-7}\ldots\ldots 3a}{\overline{t-1}.\overline{t-5}\ldots\ldots 5z}}\cdot\frac{MN}{a} - \frac{\overline{t-3}.\overline{t-7}\ldots\ldots 3zzdz}{\overline{t-1}.\overline{t-5}\ldots\ldots 5MN}.$$

Si vero t fuerit numerus pariter par, orietur formula

$$\frac{a^{t+2} dz}{z^t MN} = D\,\overline{\frac{a^{t-1}}{\overline{t-1}.z^{t-1}} \pm \frac{\overline{t-3}.a^{t-5}}{\overline{t-1}.\overline{t-5}.z^{t-5}} + \frac{\overline{t-3}.\overline{t-7}.a^{t-9}}{\overline{t-1}.\overline{t-5}.\overline{t-9}.z^{t-9}}\ldots\ldots}$$

$\overline{t-3}.$

$$\ldots \frac{\overline{t-3}.\overline{t-7}\ldots 5a^3}{\overline{t-1}.\overline{t-5}\ldots 3z^3}.\frac{MN}{a}+\frac{\overline{t-3}.\overline{t-7}\ldots 5a^2dz}{\overline{t-1}.\overline{t-5}\ldots 3MN}.$$

Habes itaque universaliter integratam formulam $\frac{a^{t+2}dz}{z^tMN}$, existente t numero pari, per arcus conicarum sectionum.

Hæc methodus plurium formularum algebraicam integrationem patefaciet, quæ sæpius alia ratione ægre admodum invenitur. Primum exemplum præbeat formula $\frac{z^8-a^8}{a^2z^4MN}.dz$, existente $M=\sqrt{aa+zz}$, & $N=\sqrt{-aa+zz}$. Formula dividatur in duas hoc modo $\frac{z^4dz}{a^2MN}-\frac{a^6dz}{z^4MN}$. Primæ formulæ hæc ex dictis habetur integratio

$$S\frac{z^4dz}{a^2MN}=\frac{zMN}{3aa}+\tfrac{1}{3}S\frac{aadz}{MN}$$; secundæ vero

$$S\frac{a^6dz}{z^4MN}=\frac{a^2MN}{3z^3}+\tfrac{1}{3}S\frac{aadz}{MN}.$$ Igitur facta deductione

$$S\frac{z^4dz}{a^2MN}-\frac{a^6dz}{z^4MN}=\frac{z}{3aa}-\frac{aa}{3z^3}.MN;$$ sive

$$S\frac{z^8-a^8.dz}{a^2z^4MN}=\frac{z^4-a^4.MN}{3a^2z^3}.$$

Multo difficilior est formula, quam pro secundo exemplo propono, nempe $\frac{9a^{16}+7z^{16}}{a^4z^{10}MN}.dz$. Divide in duas hoc pacto,

$\frac{9a^{12}dz}{z^{10}MN} + \frac{7z^{6}dz}{a^{4}MN}$. Summatoria primæ ex ſuperioribus hæc habetur

$S\frac{9a^{12}dz}{z^{10}MN} = \frac{a^9}{z^9} + \frac{7a^5}{5z^5} + \frac{7 \cdot 3 \cdot a}{5z} \cdot \frac{MN}{a} - \frac{7 \cdot 3}{5} S\frac{z^2dz}{MN}$;

Summatoria ſecundæ hæc eſt

$S\frac{7z^6dz}{a^4MN} = \frac{7z^3}{5a^3} \cdot \frac{MN}{a} + \frac{7 \cdot 3}{5} S\frac{z^2dz}{MN}$. Igitur facta additione

$S\frac{9a^{12}}{z^{10}MN} + \frac{7z^6}{a^4MN} \cdot dz$, ſive $S\frac{9a^{16} + 7z^{16}}{a^4z^{10}MN} \cdot dz =$

$\frac{a^9}{z^9} + \frac{7a^5}{5z^5} + \frac{7 \cdot 3a}{5z} + \frac{7z^3}{5a^3} \cdot \frac{MN}{a}$. Quod E. I.

Monui ſupra $S\frac{dz\sqrt{-a^2+zz}}{\sqrt{a^2+z^2}} = \frac{z\sqrt{-aa+zz}}{\sqrt{aa+zz}} - AD$,

ſeu adhibitis noſtris ſpeciebus $S\frac{Ndz}{M} = \frac{zN}{M} - AD$: atqui

$S\frac{z^2dz}{MN} - S\frac{a^2dz}{MN} = S\frac{Ndz}{M}$: Ergo

$S\frac{z^2dz}{MN} - S\frac{a^2dz}{MN} = \frac{zN}{M} - AD$, quæ proinde ſola rectificatione ellypſis indigebit. Hinc diſcimus pleraſque formulas ſola rectificata ellypſi integrare.

Unicum exemplum proponam in formula

$\frac{7^2a^{10}z^2dz + 5^2a^{12}dz}{z^{10}MN}$. Dividatur de more in duas

$\frac{7^2 a^{10} dz}{z^8 MN} + \frac{5^2 a^{12} dz}{z^{10} MN}$. Primæ integratio ex dictis hæc est

$S\frac{7^2 a^{10} dz}{z^8 MN} = \frac{7a^7}{z^7} + \frac{7.5.a^3}{3z^3} . \frac{MN}{a} + \frac{7.5}{3} S\frac{a^2 dz}{MN}$; alterius vero

$S\frac{5^2 a^{12} dz}{z^{10} MN} = \frac{5^2 a^9}{9z^9} + \frac{7.5.a^5}{9z^5} + \frac{7.5.a}{3z} . \frac{MN}{a} - \frac{7.5}{3} S\frac{z^2 dz}{MN}$. Igitur facta additione habebimus

$S\frac{7^2 a^{10} z^2 dz + 5^2 a^{12} dz}{z^{10} MN} = \frac{5^2 a^9}{9z^9} + \frac{7a^7}{z^7} + \frac{7.5.a^5}{9z^5} + \frac{7.5.a^3}{3z^3} + \frac{7.5.a}{3z} . \frac{MN}{a} + \frac{7.5}{3} . S\frac{a^2 dz}{MN} - S\frac{z^2 dz}{MN}$. Pro

$\frac{7.5}{3} . S\frac{a^2 dz}{MN} - S\frac{z^2 dz}{MN}$ substitue $-\frac{7.5 zN}{3M} + \frac{7.5.AD}{3}$, & habebis integrationem quæsitam per solam rectificationem ellypsis.

Videntur hæc satis esse, ut de integratione propositæ formulæ judicare possis. Fac valeas, & geometriam, ut soles, colas, atque amplifices.

Bononiæ tertio idus Quint. 1758.

Ferrariam ad Joannem Franciscum Malfattum.

VINCENTIUS RICCATUS

VIRO NOBILI

JORDANO COMITI RICCATO

Fratri Cariſſimo

S. P. D.

PLurimas formulas, quæ rectificatis conicis ſectionibus conſtruuntur, in diſquiſitione ſecunda perduxi ad eam, quam appellavi canonicam, quamque in prima diſquiſitione per arcus ellypticos, & hyperbolicos integravi. Verum ſi ſequaris methodum, & ſubſtitutiones, quæ ibi indicantur, conſtructiones plerumque orientur longæ, atque inelegantes: ibi enim inquirebam unice formulas, quæ per arcus ſectionum conicarum conſtrui poſſent, alias ex aliis deducens, nihil de peculiari conſtructione ſollicitus. Quapropter induſtria ab analyſtis erit exercenda, ut formulas, quas rectificatis ſectionibus conicis demonſtravimus conſtrui poſſe, conſtruantur reapſe non ineleganter. Itaque opportunum judico, aliquot exempla ad te mittere, quæ oſtendant, quomodo calculi longi ſæpe vitentur, & ad conſtructionem brevi, ac ſimplici methodo deveniatur.

Propoſuiſti mihi jampridem, ſi tenes memoria, formulam $\frac{dx.\sqrt{a+x}}{2\sqrt{x}.\overline{b+x}^{\frac{3}{2}}}$, cui addidi diviſorem 2, ut elegantiæ ſervirem. Statim ac in formulam intendi oculos, cognovi, contineri in illis, quas in ſecunda diſquiſitione docui integrari per arcus ſectionum conicarum. Quare diſquiſitionem evolvens inveni, eam pertinere ad N. XIX. Verum methodum ſequutus, quam ibi uſurpavi, adeo longum moleſtumque calculum offendi, ut piguerit, te per tam implicitas ambages deducere. Quare dedi operam, ut methodo faciliori formulam perducerem ad rectificationem ellypſis; integratur enim ſola ellypſi rectificata. Illud

autem

autem in hoc studio expertus sum, quod sæpe accidit, ut quod per methodos generales laborem poscit improbum, alia ratione obtineatur per quam facillime.

Primum multiplico formulam per constantem $b\sqrt{b}$, ut dimensionem obtineat linearem. Fit autem $\frac{b\sqrt{b}.dx.\sqrt{a+x}}{2\sqrt{x}.\overline{b+x}^{\frac{3}{2}}}$.

Tum utor non necessaria, tamen commoda substitutione $x=\frac{zz}{a}$, ut evadat $\frac{b\sqrt{ab}.dz\sqrt{aa+zz}}{\overline{ab+zz}^{\frac{3}{2}}}$.

Formulæ ita præparatæ quadratum divido in duo hoc modo

$$\frac{a^3b^3dz^2}{\overline{ab+zz}^3}, \quad \frac{ab^3zzdz^2}{\overline{ab+zz}^3}.$$

Utriusque quadrati radices accipio

$$\frac{ab.\sqrt{ab}.dz}{\overline{ab+zz}^{\frac{3}{2}}}, \quad \frac{-b\sqrt{ab}.zdz}{\overline{ab+zz}^{\frac{3}{2}}}.$$

Alteri radici præfigo signum —, quia simplicitatem calculi juvat. Utraque autem integrabilis est algebraice, & integrata exhibet

$$\frac{\sqrt{ab}.z}{\sqrt{ab+zz}}, \quad \frac{b\sqrt{ab}}{\sqrt{ab+zz}}.$$

Itaque si construas curvam, cujus abscissæ $=\frac{b\sqrt{ab}}{\sqrt{ab+zz}}$, & ordinatæ $=\frac{\sqrt{ab}.z}{\sqrt{ab+zz}}$, elementum arcus curvilinei

$$=\frac{b\sqrt{ab}.dz\sqrt{aa+zz}}{\overline{ab+zz}^{\frac{3}{2}}}.$$

Porro videamus, quænam sit hujusmodi curva. Pono $\frac{b\sqrt{ab}}{\sqrt{ab+zz}}=x$, & $\frac{\sqrt{ab}.z}{\sqrt{ab+zz}}=y$. Divido secundam æqua-

tionem per primam, & nascitur $\frac{z}{b} = \frac{y}{x}$: Ergo $z = \frac{by}{x}$.
Quo valore substituto in æquatione prima habeo
$\frac{bx\sqrt{a}}{\sqrt{ax^2+by^2}} = x$, sive $ab^2 = ax^2 + by^2$, sive $a.\overline{b^2 - x^2} = by^2$:
ergo $\overline{b^2 - x^2} : y^2 :: bb : ab$. Quæ æquatio est ad ellypsim, cujus semiaxis unus $= b$, alter $= \sqrt{ab}$.

Describamus itaque ellypsim hanc AEB, (*Fig.* 9.) cujus semiaxis $CA = b$, alter $CB = \sqrt{ab}$. Deinde sumamus
$CI = x = \frac{b\sqrt{ab}}{\sqrt{ab+zz}}$, ductaque ordinata IE, habebimus
arcum $AE = S\frac{b\sqrt{ab}.dz\sqrt{aa+zz}}{\overline{ab+zz}^{\frac{3}{2}}}$. Sed quænam est in curva linea $= z$. Invenimus supra $\frac{z}{b} = \frac{y}{x}$: Ergo $x : y :: b : z$, sive $CI : IE :: CA : z$. Quare excitata tangente AD, agatur CE, quæ producta secet tangentem in F, erit $AF = z$. Ergo secta $AF = z$, habebimus arcum $AE = S\frac{b\sqrt{ab}.dz\sqrt{aa+zz}}{\overline{aa+zz}^{\frac{3}{2}}}$, qua constructione nihil est elegantius.

Simili methodo ad integrationem perducam formulas
$\frac{b\sqrt{ab}.dz.\sqrt{aa+zz}}{\overline{-ab+zz}^{\frac{3}{2}}}$, $\frac{b\sqrt{ab}.dz.\sqrt{aa+zz}}{\overline{ab-zz}^{\frac{3}{2}}}$, a quibus dependent duæ $\frac{dx\sqrt{a+x}}{2\sqrt{x}.\overline{-b+x}^{\frac{3}{2}}}$, $\frac{dx\sqrt{a+x}}{2\sqrt{x}.\overline{b-x}^{\frac{3}{2}}}$, atque clarissime demonstrabo, earum summatoriam per arcum hyperbolicum exhiberi. Quum autem methodus sit prorsus eadem, ne molestia

stia te afficiam, calculum omitto, & solam constructionem expono.

Prima ita construitur. Statuo ad angulos (*Fig.* 10.) rectos $CA = b$, $CB = \sqrt{ab}$, positoque primo semiaxe CB, secundo CA describatur hyperbola BEN. Tum ducta AFD primo axi parallela, agatur quælibet CEF. Posita $AF = z$, erit arcus

$$BE = S \frac{-b\sqrt{ab}.\, dz \sqrt{aa + zz}}{\overline{-ab + zz}^{\frac{3}{2}}}.$$

Abscinde $AM = CB$. Si z sit minor AM, summatoria imaginaria est; si $z = AM$, summatoria infinita est; tum crescente z summatoria decrescit.

Altera formula hanc constructionem recipit. Secta $CA = b$, & ei normali $CB = \sqrt{ab}$ describo (*Fig.* 11.) hyperbolam AEN, cujus vertex sit A. Ex hoc puncto excito tangentem AD. Duco quamlibet CFE. Existente $AF = z$, erit arcus

$$AE = S \frac{b\sqrt{ab}.\, dz.\, \overline{aa + zz}^{\frac{1}{2}}}{\overline{ab + zz}^{\frac{3}{2}}},$$

quam summatoriam fieri infinitam apparet, si $z = CM = \sqrt{ab}$.

Non sum nescius, methodum hanc nullum locum habere, si alteruter ex terminis aa, zz, qui eidem radici subjciuntur, signo $-$ afficeretur. Verum in his quoque casibus methodum tibi aperiam multo faciliorem illa, quam usurpavi in secunda disquisitione. Methodum hanc exponam in formula

$$\frac{b\sqrt{ab}.\, dz \sqrt{aa - zz}}{\overline{ab + zz}^{\frac{3}{2}}}.$$

Accipio differentiam quantitatis $\frac{\sqrt{aa - zz}}{z\sqrt{ab + zz}}$, atque æquationem hanc nanciscor

$$D \frac{\sqrt{aa - zz}}{z\sqrt{ab + zz}} = \frac{-dz\sqrt{aa - zz}}{zz\sqrt{ab + zz}} - \frac{dz}{\sqrt{aa - zz}.\, \sqrt{ab + zz}}$$

$-\frac{dz\sqrt{aa-zz}}{\overline{ab+zz}^{\frac{3}{2}}}$; sive translatis opportune terminis, & duobus ad eamdem denominationem redactis, factaque multiplicatione per $b\sqrt{ab}$, acceptaque summatoria

$$S\frac{b\sqrt{ab}.dz\sqrt{aa-zz}}{\overline{ab+zz}^{\frac{3}{2}}}=\frac{-b\sqrt{ab}.\sqrt{aa-zz}}{z\sqrt{ab+zz}}-$$

$$S\frac{a^2b\sqrt{ab}.dz}{z^2\sqrt{aa-zz}.\sqrt{ab+zz}}.$$

Habeo ex N. VIII secundæ disquisitionis, formulam

$\frac{dz}{zz\sqrt{aa-zz}.\sqrt{ab+zz}}$ æquare

$$-D\frac{\sqrt{aa-zz}.\sqrt{ab+zz}}{a^3bz}-\frac{zzdz}{a^3b\sqrt{aa-zz}.\sqrt{ab+zz}}: \text{ ergo}$$

$$S\frac{b\sqrt{ab}.dz\sqrt{aa-zz}}{\overline{ab+zz}^{\frac{3}{2}}}=\frac{-b\sqrt{ab}.\sqrt{aa-zz}}{z\sqrt{ab+zz}}+$$

$$\frac{\sqrt{b}}{\sqrt{a}}.\frac{\sqrt{aa-zz}.\sqrt{ab+zz}}{z}+S\frac{\sqrt{b}}{\sqrt{a}}.\frac{z^2dz}{\sqrt{aa-zz}.\sqrt{ab+zz}}.$$ Redactis vero duobus terminis algebraicis ad eumdem denominatorem, fiet

$$S\frac{b\sqrt{ab}.dz\sqrt{aa-zz}}{\overline{ab+zz}^{\frac{3}{2}}}=\frac{\sqrt{b}}{\sqrt{a}}.\frac{z\sqrt{aa-zz}}{\sqrt{ab+zz}}+$$

$$S\frac{\sqrt{b}}{\sqrt{a}}.\frac{z^2dz}{\sqrt{aa-zz}.\sqrt{ab+zz}}.$$

Habeo item ex N. III ejusdem disquisitionis

$$\frac{zzdz}{\sqrt{aa-zz}\,.\,\sqrt{ab+zz}}=\frac{-bdz\sqrt{aa-zz}}{a+b\,.\,\sqrt{ab+zz}}+\frac{adz\sqrt{ab+zz}}{a+b\,.\,\sqrt{aa-zz}}:$$

ergo opportuna facta substitutione invenio

$$S\,\frac{b\sqrt{ab}\,.\,dz\sqrt{aa-zz}}{\overline{ab+zz}^{\frac{3}{2}}}=\frac{\sqrt{b}}{\sqrt{a}}\,.\,\frac{z\sqrt{aa-zz}}{\sqrt{ab+zz}}$$

$$-S\,\frac{b\sqrt{b}}{a+b\,.\,\sqrt{a}}\,.\,\frac{dz\sqrt{aa-zz}}{\sqrt{ab+zz}}+S\,\frac{\sqrt{ab}}{a+b}\,.\,\frac{dz\sqrt{ab+zz}}{\sqrt{aa-zz}}.$$

Formula $\frac{dz\sqrt{aa-zz}}{\sqrt{ab+zz}}$ ex N. XIV primæ disquisitionis integrationem recipit per arcus ellypticos, & hyperbolicos. Hujusmodi autem nascitur constructio. Describatur ellypsis, cujus semiaxis major C A $=\sqrt{aa+ab}$, (*Fig.* 12.) minor C B $=\sqrt{ab}$: abscinde C G $=\frac{z\sqrt{b}}{\sqrt{a}}$, & determina punctum D. Tum describe hyperbolam, cujus semiaxis primus C B $=\sqrt{ab}$, secundus C M $=b$, & seca C P $=\frac{a\sqrt{b}}{\sqrt{a+b}}\,.\,\frac{\sqrt{ab+zz}}{z}$, & determina arcum B N : erit

$$S\,\frac{dz\sqrt{aa-zz}}{\sqrt{ab+zz}}=\frac{a+b}{b}\,.\,\frac{\sqrt{ab+zz}\,.\,\sqrt{aa-zz}}{z}-\text{B D}.\frac{a+b}{b}$$

$$-\text{A D}-\text{B N}\,.\,\frac{a+b}{b}.$$

Quoniam B D + A D æquat quadrantem ellypticum, qui constans est, omitti potest; in integratione enim ea constans addenda est, quam circumstantiæ requirunt: igitur

$$S\,\frac{b\sqrt{ab}}{a+b\,\sqrt{a}}\,.\,\frac{dz\sqrt{aa-zz}}{\sqrt{ab+zz}}=\frac{\sqrt{b}}{\sqrt{a}}\,.\,\frac{\sqrt{aa-zz}\,.\,\sqrt{ab+zz}}{z}$$

$$-\frac{\sqrt{ab}}{a+b}\,\text{B D}-\frac{\sqrt{b}}{\sqrt{a}}.\text{B N}.$$

Al-

Altera formula $\frac{dz\sqrt{ab+zz}}{\sqrt{aa-zz}}$ ex N. II disquisitionis primæ per solam ellypsim construitur: imo eadem ellypsis ADB adhibenda est, in qua pariter sumenda $CG=\frac{z\sqrt{b}}{\sqrt{a}}$: & habetur

$$S\frac{dz\sqrt{ab+zz}}{\sqrt{aa-zz}}=AD.$$ Igitur

$$S\frac{\sqrt{ab}}{a+b}\cdot\frac{dz\sqrt{ab+zz}}{\sqrt{aa-zz}}=\frac{\sqrt{ab}}{a+b}\cdot AD.$$ Quapropter

$$Sb\sqrt{ab}\cdot\frac{dz\sqrt{aa-zz}}{\overline{ab+zz}^{\frac{3}{2}}}=\frac{\sqrt{b}}{\sqrt{a}}\cdot\frac{z\sqrt{aa-zz}}{\sqrt{ab+zz}}-\frac{\sqrt{b}}{\sqrt{a}}\cdot\frac{\sqrt{aa-zz}\cdot\sqrt{ab+zz}}{z}$$

$$+\frac{\sqrt{ab}}{a+b}\cdot BD+\frac{\sqrt{ab}}{a+b}\cdot AD+\frac{\sqrt{b}}{\sqrt{a}}\cdot BN:$$ sive omisso ellyptico quadrante, & redactis duobus terminis algebraicis ad eamdem denominationem

$$Sb\sqrt{ab}\cdot\frac{dz\sqrt{aa-zz}}{\overline{ab+zz}^{\frac{3}{2}}}=\frac{-\sqrt{b}}{\sqrt{a}}\cdot\frac{ab\sqrt{aa-zz}}{z\sqrt{ab+zz}}+\frac{\sqrt{b}}{\sqrt{a}}\cdot BN,$$

quæ per solam hyperbolæ rectificationem, ut patet, integratur. Similis methodus in aliis quoque formulis resolvendis afferet utilitatem.

Eodem modo tractare licet formulam latius patentem, nempe

$\frac{z^{t}dz\sqrt{aa-zz}}{\overline{ab+zz}^{\frac{3}{2}}}$, in qua numerus t sit par vel positivus, vel negativus. Namque accipio differentiam formulæ $\frac{z^{r}\sqrt{aa-zz}}{\sqrt{ab+zz}}$, ut sit

$$D\frac{z^{r}\sqrt{aa-zz}}{\sqrt{ab+zz}}=\frac{rz^{r-1}dz\sqrt{aa-zz}}{\sqrt{ab+zz}}-\frac{z^{r+1}dz}{\sqrt{aa-zz}\cdot\sqrt{ab+zz}}-$$

$$\frac{z^{r+1}dz\sqrt{aa-zz}}{\overline{ab+zz}^{\frac{3}{2}}}.$$ Quare transpositis terminis, & duobus ad eam-

eamdem denominationem redactis, sumptaque summatoria
$$S\frac{z^{r+1}dz\sqrt{aa-zz}}{\overline{ab+zz}^{\frac{3}{2}}}=-\frac{z^{r}\sqrt{aa-zz}}{\sqrt{ab+zz}}+$$
$$S\frac{raaz^{r-1}dz-(\overline{r+1}.z^{r+1}dz}{\sqrt{aa-zz}.\sqrt{ab+zz}}.$$ Has autem formulas
$$\frac{z^{r-1}dz}{\sqrt{aa-zz}.\sqrt{ab+zz}},\ \frac{z^{r+1}dz}{\sqrt{aa-zz}.\sqrt{ab+zz}}$$ per methodum explicatam N. VII, & VIII secundæ disquisitionis ad canonicas perduces, si numeri $r-1$, $r+1$ sint pares vel affirmativi, vel negativi. Quæ methodus locum habebit, tametsi aliis signis afficiantur termini, qui subsunt potestatibus fractis. Sed de his formulis satis.

Transeo ad aliam, quæ mihi non ita pridem proposita est, nempe $\frac{dx.\overline{aa+xx}^{\frac{1}{3}}}{a^{\frac{2}{3}}}$. Hæc in secunda disquisitione continetur N. XLI; sed facilius ad rectificationem conicarum sectionum hac methodo perducetur. Constructam intelligo (*Fig.* 13.) curvam EQ, cujus abscissæ KH $=x$, ordinatæ HQ $=a^{\frac{1}{3}}.\overline{aa+xx}^{\frac{1}{3}}$. Manifestum est $S\frac{dx.\overline{aa+xx}^{\frac{1}{3}}}{a^{\frac{2}{3}}}=\frac{KEQH}{a}$. Curva EQ est species quædam hyperbolæ gradus superioris, cujus progressus tibi cognitus est. Iam vero, existente EK $=a$, voca EM $=t$: Ergo habebimus $a+t=a^{\frac{1}{3}}.\overline{a^2+x^2}^{\frac{1}{3}}$, sive
$$a^3+3aat+3att+t^3=a^3+ax^2:\text{ ergo}$$
$$\frac{\sqrt{t}.\sqrt{3aa+3at+tt}}{\sqrt{a}}=x$$

Spatium KEQH = rectan. KMQH − EMQ: atqui

$$EMQ = S\, x\, dt = S\frac{dt\sqrt{t}\,.\sqrt{3a^2+3at+tt}}{\sqrt{a}}:\ \text{ergo}$$

$$S\,\frac{dx\,.\,\overline{aa+xx}^{\frac{1}{3}}}{a^{\frac{2}{3}}} = \frac{x\,.\,\overline{a+t}}{a} - S\,\frac{dt\sqrt{t}\,.\sqrt{3aa+3at+tt}}{a\sqrt{a}}.$$

Igitur qui constructionem dederit formulæ $\frac{dt\sqrt{t}\,.\sqrt{3aa+3at+tt}}{a\sqrt{a}}$, idem & propositam formulam facili negotio ad constructionem deducet.

Ut voti compos fiam, formulam $\frac{dt\sqrt{t}\,.\sqrt{3aa+3at+tt}}{a\sqrt{a}}$, vocata $\sqrt{3aa+3at+tt} = M$, distribuo in tres hoc modo

$$\frac{M\,dt\sqrt{t}}{a\sqrt{a}} = \frac{3dt\sqrt{at}}{M} + \frac{3t\,dt\sqrt{t}}{M\sqrt{a}} + \frac{t^2dt\sqrt{t}}{aM\sqrt{a}}:\ \text{atqui}$$

$$\frac{t^2dt\sqrt{t}}{aM\sqrt{a}} = \frac{2}{5}D\frac{Mt\sqrt{t}}{a\sqrt{a}} - \frac{12}{5}.\frac{t\,dt\sqrt{t}}{M\sqrt{a}} - \frac{9}{5}.\frac{dt\sqrt{at}}{M}:$$

ergo fiet

$$\frac{M\,dt\sqrt{t}}{a\sqrt{a}} = \frac{2}{5}D\frac{Mt\sqrt{t}}{a\sqrt{a}} + \frac{3}{5}\frac{t\,dt\sqrt{t}}{M\sqrt{a}} + \frac{6}{5}.\frac{dt\sqrt{at}}{M}:\ \text{atqui}$$

$$\frac{3}{5}\frac{t\,dt\sqrt{t}}{M\sqrt{a}} = \frac{2}{5}D\frac{M\sqrt{t}}{\sqrt{a}} - \frac{6}{5}.\frac{dt\sqrt{at}}{M} - \frac{3}{5}.\frac{a\,dt\sqrt{a}}{M\sqrt{t}}.$$

ergo formula in hanc mutabitur

$$\frac{M\,dt\sqrt{t}}{a\sqrt{a}} = \frac{2}{5}D\frac{\overline{a+t}\,M\sqrt{t}}{a\sqrt{a}} - \frac{3}{5}\frac{a\,dt\sqrt{a}}{M\sqrt{t}}.$$

Hanc formulam nacti advertamus $\frac{\overline{a+t}\,.M\sqrt{t}}{a\sqrt{a}}$ nihil aliud

esse

esse, quam rectangulum KMQH divisum per a: nam $KM = a + t$, & $KH = x = \frac{M\sqrt{t}}{\sqrt{a}}$: igitur si ex duabus quintis partibus rectanguli KMQH auferam tres quintas partes $S\frac{adt\sqrt{a}}{M\sqrt{t}}$, habeo $S\frac{Mdt\sqrt{t}}{a\sqrt{a}}$. Quare eo res redacta est, ut inveniatur $S\frac{adt\sqrt{a}}{M\sqrt{t}}$, quæ, posita $t = o$, nullescat.

Utor jam necessaria substitutione $y = \sqrt{at + \frac{3aa}{2} + a\sqrt{3aa + 3at + tt}}$, ex qua oritur (A) $y^2 - at - \frac{3aa}{2} = a\sqrt{3aa + 3at + tt}$, & iterum elevando ad secundam potestatem

$$y^4 - 2y^2 . at + \frac{3}{2}aa + a^2t^2 + 3a^3t + \frac{9}{4}a^4 = 3a^4 + 3a^3t + a^2t^2,$$

& deletis delendis $y^4 - 2y^2 . at + \frac{3}{2}aa - \frac{3}{4}a^4 = o$, sive (B) $\frac{y^4 - \frac{3}{4}a^4}{2y^2} = at + \frac{3}{2}a^2$: quo valore translato in æquationem A fit (C) $\frac{y^4 + \frac{3}{4}a^4}{2y^2} = a\sqrt{3a^2 + 3at + tt}$. Præterea differentiata æquatione B oritur $\frac{dy}{y^3} . y^4 + \frac{3}{4}a^4 = adt$. Hæc dividatur per C, & nascetur $\frac{2dy}{y} = \frac{dt}{\sqrt{3aa + 3at + tt}}$. Demum formula B exhibet $\frac{y^4 - 3aayy - \frac{3}{4}a^4}{2yy} = at$, sive

$\dfrac{\overline{yy - \frac{3}{2}a^2 + aa\sqrt{3}} \cdot \overline{yy - \frac{3}{2}a^2 - aa\sqrt{3}}}{2yy} = at$. Igitur tandem fiet

$$\frac{2a^2 dy\sqrt{2}}{\sqrt{yy - \frac{3}{2}a^2 + a^2\sqrt{3}} \cdot \sqrt{yy - \frac{3a^2}{2} - a^2\sqrt{3}}} =$$

$\dfrac{adt\sqrt{a}}{\sqrt{t} \cdot \sqrt{3aa + 3at + tt}}$. Hujus vero formulæ, ad quam pervenimus, constructionem supposita hyperbolæ, & ellypsis rectificatione per ea, quæ tradita sunt in duabus disquisitionibus, facile obtinebis.

Non dissimilis methodus deteget formulas duas

$\dfrac{dx \cdot \overline{a^2 - x^2}^{\frac{1}{3}}}{a^{\frac{2}{3}}}$, $\dfrac{dx \cdot \overline{xx - aa}^{\frac{1}{3}}}{a^{\frac{2}{3}}}$ dependere a formula

$\dfrac{dt\sqrt{t} \cdot \sqrt{3aa - 3at + tt}}{a\sqrt{a}}$, quæ facilius construitur per sectionum conicarum rectificationem. Namque in prima utere substitutione $a - t = a^{\frac{1}{3}} \cdot \overline{aa - xx}^{\frac{1}{3}}$, qua elevata ad tertiam potestatem habebis $a^3 - 3aat + 3att - t^3 = a^3 - ax^2$: ergo $x = \dfrac{\sqrt{t} \cdot \sqrt{3aa - 3at + tt}}{\sqrt{a}}$. Quoniam vero

$S\,\dfrac{\overline{a - t} \cdot dx}{a} = \dfrac{x \cdot \overline{a - t}}{a} + S\,x\,dt$ habebimus

$$S\frac{dx.\overline{aa-xx}^{\frac{1}{3}}}{a^{\frac{2}{3}}}=\frac{x.\overline{a-t}}{a}+S\frac{dt\sqrt{t}.\sqrt{3aa-3at+tt}}{a\sqrt{a}}.$$

In secunda formula substitutio $t-a=a^{\frac{1}{3}}.\overline{xx-aa}^{\frac{1}{3}}$ eodem modo te deducet ad æquationem

$$S\frac{dx.\overline{xx-aa}^{\frac{1}{3}}}{a^{\frac{2}{3}}}=\frac{x.\overline{t-a}}{a}-S\frac{dt\sqrt{t}.\sqrt{3a^2-3at+tt}}{a\sqrt{a}}.$$

Verum, omissis tamdem his formulis, solutionem præbeamus problematis sane veteris, quod tamen nulla adhuc constructione oculis subjectum est. Problema est hujusmodi. Invenire tempus descensus penduli circularis. Sit semicirculus CEA, cujus (*Fig.* 14.) centrum K. Descendat mobile ex punito quietis E per arcum circularem ELC: quæritur tempus descensus per quemlibet arcum EL. Agantur horizontales EG, LR, & sumantur analoga elementa Ll, Rr. Vocetur radius circuli $=r$, $CG=a$, $CR=x$: igitur $GR=a-x$. Omissa gravitate acceleratrice, quæ supponitur semper eadem, tempusculum per spatiolum Ll exprimitur a formula $\frac{-rdx}{\sqrt{x}.\sqrt{2r-x}.\sqrt{a-x}}$ quæ, ut ad linearem potestatem redigatur, multiplicetur per $\sqrt{2r}$; & fiet

$$\frac{-r\sqrt{2r}.dx}{\sqrt{x}.\sqrt{2r-x}.\sqrt{a-x}}.$$

Duco cordam CL, quam voco $=z$, & erit $2rx=zz$. Itaque facta substitutione formula exprimens tempusculum in hanc mutabitur $\frac{-4rrdz}{\sqrt{4rr-zz}.\sqrt{2ra-zz}}$. Duco cordam CE, eamque voco $=b$, ut sit $2ra=bb$. Ergo tempusculum exprimetur a formula $\frac{-4rrdz}{\sqrt{4rr-zz}.\sqrt{bb-zz}}$.

Hæc formula, ut constat ex N. X primæ disquisitionis, ita resolvitur in duas

$-4rrdz$

$$\frac{-4rrdz}{\sqrt{4rr-zz}.\sqrt{bb-zz}} = \frac{-4rrdz\sqrt{4rr-zz}}{4rr-bb.\sqrt{bb-zz}} +$$

$\frac{4rrdz\sqrt{bb-zz}}{4rr-bb.\sqrt{4rr-zz}}$. Ducatur corda AE: manifestum est, hujus quadratum esse differentiam duorum quadratorum CA, CE: ergo, vocata $AE = c$, erit $cc = 4rr - bb$: igitur

$$\frac{-4rrdz}{\sqrt{4rr-zz}.\sqrt{bb-zz}} = \frac{-4rrdz\sqrt{4rr-zz}}{cc\sqrt{bb-zz}} +$$

$$\frac{4rrdz\sqrt{bb-zz}}{cc\sqrt{4rr-zz}}.$$

Prima ex his formulis pertinet ad rectificationem folius ellypsis, & construitur hoc modo. Normalis diametro excitetur $CB = AE$: atque positis semiaxibus AC, CB describatur ellypsis AB. Tum fiat ubique $CE = b : CA = 2r :: CL = z : CF = \frac{2rz}{b}$. Advertendum est autem, punctum F coincidere cum A, si CL coincidat cum CE. Ordina in ellypsi rectam FD. His positis habebimus $S\frac{-dz\sqrt{4rr-zz}}{\sqrt{bb-zz}} = AD$, quæ evanescit facta $z = CE = b$, sive $x = a$. Igitur

$$S\frac{-4rrdz\sqrt{4rr-zz}}{cc\sqrt{bb-zz}} = \frac{4rr}{cc} AD.$$

Secunda ex præmissis formulis integratur per rectificationem hyperbolæ ex N. IX primæ disquisitionis. Integratio autem, quæ ibi traditur, est hujusmodi. Cum eodem semiaxe primo AC, & cum semiaxe secundo $CM = \frac{2rc}{b}$, quæ est quarta proportionalis post CE, AE, AC, describatur hyperbola AVO, & sumpta abscissa $CP = \frac{b\sqrt{4rr-zz}}{\sqrt{bb-zz}}$, ductaque ordinadinata PN, erit $S\frac{dz\sqrt{bb-zz}}{\sqrt{4rr-zz}}=\frac{z\sqrt{4rr-zz}}{\sqrt{bb-zz}}-\frac{b}{2r}.AN$. Quum autem hæc summatoria debeat evanescere facta $z=b$, necesse esset addere, & demere constantes infinitas; nam posita $z=b$, tam abscissa CP, quam quantitas algebraica $\frac{z\sqrt{4rr-zz}}{\sqrt{bb-zz}}$ evadit infinita.

Ut hoc incommodum vitem, utor artificio alias usurpato: nimirum determino eos arcus in hyperbola, quorum differentia rectificabilis est. Hanc ob rem abscindenda constans $CT=\sqrt{4rr+2rc}$, tum determinandus arcus constans AV, deinde secanda $CS=\frac{2rb}{z}$. Erit ex litteris ad Mariscottum, & ex prima disquisitione
$\frac{2rb\sqrt{4rr-zz}}{z\sqrt{bb-zz}}-\frac{2r}{b}.\overline{2r+c}+2AV-AO=AN$: quo valore substituto in formula superiore, habebimus
$S\frac{dz\sqrt{bb-zz}}{\sqrt{4rr-zz}}=\frac{-\sqrt{4rr-zz}.\sqrt{bb-zz}}{z}+2r+c-\frac{b}{r}AV+\frac{b}{2r}.AO$. Quum autem hæc summatoria debeat nullescere facta $z=b$, addatur, oportet, in integratione $\frac{b}{r}AV-2r-c$; nam cætera omnia evanescunt: igitur fiet
$S\frac{dz\sqrt{bb-zz}}{\sqrt{4rr-zz}}=\frac{-\sqrt{4rr-zz}.\sqrt{bb-zz}}{z}+\frac{b}{2r}AO$. Quapropter tempus per arcum EL exprimetur a formula
$\frac{4rr}{cc}AD-\frac{4rr}{cc}.\frac{\sqrt{4rr-zz}.\sqrt{bb-zz}}{z}+\frac{2rb}{cc}.AO$.

Quæ formula usui esse potest, donec non sit $z=o$. Verum si quæ-si quæramus tempus per integrum arcum E C, tam quantitas algebraica, quam arcus A O evadit infinitus. Quare, determinata jam constante addenda in integratione, iterum pro arcu A O substituo A N, ut habeam tempus per arcum E L hoc modo expressum.

$$\frac{4rr}{cc}.AD + \frac{4rr}{cc}.\frac{z\sqrt{4rr-zz}}{\sqrt{bb-zz}} - \frac{4rr}{cc}.\overline{2r+c} +$$

$$\frac{4rb}{cc}.AV - \frac{b}{2r}.AN.$$ Igitur facta $z=o$, habetur tempus per integrum arcum E C hoc modo expressum

$$\frac{4rr}{cc}.AB - \frac{4rr}{cc}.\overline{2r+c} + \frac{4rb}{cc}.AV.$$

Ex hac formula, si sumatur b infinite exigua, ut oscillatio fiat per arcum minimum, & proxime $c=2r$, probabitur circularis isochronismus. Nam in hac hypothesi quantitas algebraica $\frac{4rr}{cc}.\overline{2r+c} = \frac{4rb}{cc}.AV$. Ergo ex contrarietate signorum eliduntur. Æqualitatem autem ita probo. Quantitas $\frac{4rr}{cc}.\overline{2r+c}$ fit proxime $=4r$: sed etiam $\frac{4rb}{cc}.AV$ fit $=4r$. Etenim hyperbolæ semiaxis secundus CM positus est $=\frac{2rc}{b}$: ergo posita b infinite exigua infinitus est: ergo hyperbola ANV convertitur in lineam rectam parallelam secundo axi CM: ergo definitus arcus A V fiet æqualis ejus ordinatæ T V. Inveniamus generatim hanc ordinatam. Ex natura hyperbolæ $CA^2 : CM^2 :: CT^2 - CA^2 : TV^2$, sive analitice $4rr : \frac{4rrcc}{bb} :: 4rr + 2rc - 4rr : TV^2 = \frac{2rc^3}{b^2}$: Ergo $TV = \frac{c\sqrt{2rc}}{b}$: ergo posita b infinitesima etiam arcus $AV = \frac{c\sqrt{2rc}}{b}$: ergo

$\frac{4rb}{cc}$ A V $= \frac{4rc\sqrt{2rc}}{cc}$, & facta $c = 2r$, erit $\frac{4rb}{cc}$ A V $= 4r$.

Quapropter tempus per arcum minimum E C repræsentatur a formula $\frac{4rr}{cc}$ A B, seu existente $c = 2r$ a solo quadrante ellyptico A B. Atqui semiaxis ellypsis C B $= c$ fit proxime $= 2r$, hoc est semiaxi primo C A: ergo tempus oscillationis minimæ per E C exprimitur a quadrante circuli, qui habet radium duplum ejus, in quo fit oscillatio. Quare oscillationes vel majores sint, vel minores, dummodo infinitesimæ, in circulo sunt isochronæ.

Si aveas comparare tempus descensus per minimum arcum E C cum tempore motus rectilinei, oportet formulam $\frac{dx}{\sqrt{x}}$ exprimentem tempusculum motus verticalis multiplicare per $\sqrt{2r}$, ut fiat analoga illi, quam adhibuimus in circulo. Exprimetur itaque tempusculum in motu verticali per formulam $\frac{dx\sqrt{2r}}{\sqrt{x}}$: ergo facta integratione tempus motus verticalis per spatium x erit $= 2\sqrt{2rx}$: ergo tempus motus per spatium x erit ad tempus per minimum arcum E C, ut $2\sqrt{2rx}$ ad quadrantem A B. Si $x = r$, habebimus tempus descensus per radium K C ad tempus per arcum E C minimum, ut duplex corda quadrantis circuli A E C ad quadrantem A B, aut ad semicirferentiam A E C, sive ut corda quadrantis ad quadrantem. Si $x = \frac{r}{2}$, tempus per dimidium radii, quod invenitur $= 2r$, erit ad tempus per minimum arcum E C, ut diameter ad semicircumferentiam, seu ut radius ad quadrantem. Demum si $x = 2r$, tempus descensus rectilinei prodit $= 4r$: ergo tempus descensus per diametrum aut per quamlibet cordam erit ut dupla diameter ad semicircumferentiam, sive ut diameter ad quadrantem. Sed satis de pendulo circulari.

Verum, antequam epistolæ finem facio, per te mihi liceat,

 ad-

addere determinationem temporis in pendulo parabolico ope constructionis, quæ sua se simplicitate commendat. In Parabola CLE descendat mobile ex (*Fig.* 15.) puncto quietis E: quæritur tempus descensus per quemlibet arcum EL. Parameter CB $= a$, CF $= b$, FE $= c$, ut sit $ab = cc$: CR $= x$;

erit elementum Ll $= \dfrac{-dx\sqrt{x+\frac{a}{4}}}{\sqrt{x}}$: ergo tempus per hoc

elementum exprimetur a formula $\dfrac{-dx\sqrt{x+\frac{a}{4}}}{\sqrt{x}\,.\,\sqrt{b-x}}$. Hæc ut ad

linearem dimensionem redigatur, multiplicetur per $\sqrt{a}$, ut tempusculum exprimatur a formula $\dfrac{-dx\sqrt{a}\,.\,\sqrt{a+4x}}{2\sqrt{x}\,.\,\sqrt{b-x}}$.

Quoniam, vocata RL $= z$, est $ax = zz$, & $\sqrt{x} = \dfrac{z}{\sqrt{a}}$, &

$\dfrac{dx}{2\sqrt{x}} = \dfrac{dz}{\sqrt{a}}$, peracta substitutione formula exprimens tempusculum per Ll fiet $\dfrac{-dz\sqrt{aa+4zz}}{\sqrt{ab-zz}}$, sive posito cc pro

ab, $\dfrac{-dz\sqrt{aa+4zz}}{\sqrt{cc-zz}}$. Quæ formula, ut discimus ex N. II primæ disquisitionis, integratur sola ellypsi rectificata in hunc modum. Produc FC in A, donec CA $= \sqrt{aa+4cc}$; atque hic erit semiaxis major ellypsis; minor autem sit CB $= a$, hoc est Parabolæ parametro. Sumatur in minore abscissa CG $= \dfrac{az}{c}$, tempus per EL exprimetur ab arcu BD, per LC ab arcu DA, & tempus descensus per integrum arcum EC a quadrante ellyptico BA.

Si CF $= b$, adeoque FE $= c$ sit infinite parva, semiaxis ma-

major CA fit per adæquationem æqualis minori CB, adeoque quadrans circuli metitur tempus per minimum arcum EC, quicumque demum hic sit vel major, vel minor. Quare etiam in parabola pendulum minimas oscillationes complens erit isochronum.

Hæc tibi scribenda censui, ut utilitas integrationis per arcus ellypticos, & hyperbolicos magis magisque patefiat. Te, Fratres reliquos, Fratrisque nostri Uxorem valere jubeo.

Bononiæ pridie Nonas Januarii 1759.

Tarvisium ad Comitem Jordanum Riccatum.

VINCENTIUS RICCATUS

JACOBO MARISCOTTO

Geographiæ, & Nauticæ Professori

S. P. D.

SPem omnem, si qua in te, Vir Clarissime, reliqua est, construendi formulam illam, quæ tibi in rectificanda nescio qua curva sese obtulit, per quadraturas Circuli, & hyperbolæ, jubeo te omnino deponere; ea enim non construetur, nisi supposita etiam conicarum sectionum rectificatione. Antequam tamen id tibi ex methodis a me non ita pridem inventis demonstro, liceat paullo attentius contemplari formulam illam tuam, & inquirere, cujus nam curvæ rectificationem exhibeat. Gratum enim tibi futurum spero, si curvam, in qua nunc versaris, ex formula rectificationis, quasi divinans ostendero.

Formulam, quam mihi proposuisti, nempe

$$\frac{dx\sqrt{a^4+3a^2x^2+x^4}}{a^2+x^2}, \text{ hoc modo dispono}$$

$$dx\sqrt{1+\frac{a^2x^2}{\overline{a^2+x^2}^2}}.$$ Hujusce formulæ quadratum divido in

hæc duo dx^2, $\frac{aaxxdx^2}{\overline{aa+xx}^2}$, quorum radices reales sunt, nempe

dx, $\frac{axdx}{aa+xx}$. Si has considerarem tamquam elementa duarum coordinatarum orthogonalium, curva oriretur transcendens, cujus constructio dependeret ab hyperbolæ quadratura, de qua, suspicor, te minime cogitare.

Quapropter curvam refero ad focum, & suppono ordinatam

FE

$FE = x$, elementum (*Fig.* 16.) autem circulare $Eb = \frac{ax\,dx}{aa + xx}$, ut curvæ elementum Ee a tua formula exprimatur. Abscindo $FZ = a$, & hoc radio describo elementum Zz, quod ut inveniam, facio $x : a :: Eb = \frac{ax\,dx}{aa + xx} : Zz = \frac{aa\,dx}{aa + xx}$: atqui hæc formula est elementum arcus circularis, cujus radius $= a$, tangens $= x$: ergo integrale elementi Zz æquale est arcui ejusdem radii, cujus tangens $= x$. Quare si ducta qualibet FB producatur arcus Zz in B, cujus arcus tangens sit ZK, debebit hæc æquare FE. Itaque hæc oritur curvæ constructio. Centro F, radio $FB = a$ describatur circulus BZC. Ducta qualibet FZ, age ZK tangentem circulum in puncto Z, cui æqualem abscinde FE, punctum E erit in curva.

Sit FH perpendicularis rectæ FB, & EH rectæ FE. Triangula duo FZK, HEF ob æqualitatem angulorum sunt prorsus similia; sed $FE = ZK$: ergo EH æquat ZF, scilicet est constans, & $= a$. Quæ maxime simplex est proprietas curvæ, ut constans sit perpendicularis ordinatæ FE terminata a recta FH. Nonne ex hac proprietate naturam curvæ scrutatus es, cujus deinde rectificationem requirens in formulam paullo ante expositam incidisti?

Sed rem propius aggrediens, formulam hac ratione dispono $dx\sqrt{1 + \frac{aaxx}{\overline{aa + xx}^2}}$. Utor substitutione $\frac{aa + xx}{x} = 2z$, quæ formulam in hanc convertit

$$dx\sqrt{1 + \frac{aa}{4zz}} = \frac{dx}{z}\sqrt{zz + \frac{aa}{4}}.$$

Ut arceatur dx invenienda est x per z, quod præstat resolutio æquationis quadraticæ $xx - 2xz = -aa$, quæ dabit $x = z \pm \sqrt{zz - aa}$. Ex hac docemur, duplicem valorem x respondere cuicumque z. Quare opportunum erit in figura valorem z determinare.

Producatur tangens KZ in I, erit $KI = \frac{aa + xx}{x} = 2z$. Agatur FA dividens bifariam angulum rectum BFC, quæ transibit fibit per interfectionem circuli BAC, & curvæ FAE, quia ob angulum femirectum AFB tangens arcus BA æqualis eft radio. Fac angulum AFD = AFZ, & linea FD producta fecet circulum in U. Manifeftum eft, tangentem circuli ductam per punctum U terminatam ad rectas FB, FC æquare KZI. Quare duplex x, quæ refpondet eidem z erit FD, FE, quarum prima $= z - \sqrt{zz - aa}$, altera $= z + \sqrt{zz - aa}$.

His determinatis, quæ elegantiam non mediocrem conciliant, neceffarium eft advertere, arcum minimum Ee, prout eft elementum arcus AE, exprimi quidem per formulam $\frac{dx}{z}\sqrt{zz + \frac{aa}{4}}$; at arcus minimus Dd, prout eft elementum arcus AD, debet exprimi per formulam figno — affectam, nempe per $-\frac{dx}{z}\sqrt{zz + \frac{aa}{4}}$, quia crefcente AD decrefcit x. Nunc vero profequens calculum differentio æquationem $x = z \pm \sqrt{zz - aa}$, & invenio $dx = dz \pm \frac{zdz}{\sqrt{zz - aa}}$, qui valor fubftitutus in formula rectificationis dabit

$$\frac{dz}{z}\sqrt{zz + \frac{aa}{4}} \pm \frac{dz\sqrt{zz + \frac{aa}{4}}}{\sqrt{zz - aa}}.$$

Quæ dicta funt fatis demonftrant, quid fignificet fignorum ambiguitas. Namque elementum $Ee = \frac{dz}{z}\sqrt{zz + \frac{aa}{4}} + \frac{dz\sqrt{zz + \frac{aa}{4}}}{\sqrt{zz - aa}}$, contra elementum $Dd = \frac{-dz}{z}\sqrt{zz + \frac{aa}{4}} + \frac{dz\sqrt{zz + \frac{aa}{4}}}{\sqrt{zz - aa}}$; anguli infinitefimi EF*e*, DF*d* æquales fumendi funt.

Ex his colligimus, differentiam arcuum

$Ee - Dd = \frac{2dz}{z}\sqrt{zz + \frac{aa}{4}}$; summam vero

$Ee + Dd = \frac{2dz\sqrt{zz + \frac{aa}{4}}}{\sqrt{zz - aa}}$. Quare facta integratione erit

$AE - AD = S\frac{2dz}{z}\sqrt{zz + \frac{aa}{4}}$, quæ, ut mox constabit, a sola quadratura hyperbolæ dependet. Præterea

$AE + AD = S\frac{2dz\sqrt{zz + \frac{aa}{4}}}{\sqrt{zz - aa}}$. Istæ summatoriæ sic accipiendæ sunt, ut posita $z = x = a$, evanescant, & nihilo æquales fiant. Hoc adverte, non posse esse $z < a$, sed ab a usque in infinitum augeri.

Jam vero per logarithmos integremus formulam $\frac{dz}{z}\sqrt{zz + \frac{aa}{4}}$, quæ ita erit disponenda

$\frac{zdz}{\sqrt{zz + \frac{aa}{4}}} + \frac{\frac{aa}{4}dz}{z\sqrt{zz + \frac{aa}{4}}}$. Primæ formulæ summa

$= \sqrt{zz + \frac{aa}{4}} - \frac{a}{2}.\sqrt{5}$; ita enim evanescit posita $z = a$.

Ut secunda ad logarithmos reducatur, adhibenda est substitutio.

$\sqrt{zz + \frac{aa}{4}} = y$, & formula in hanc mutabitur $\frac{\frac{aa}{4}dy}{yy - \frac{aa}{4}}$, quæ resolvitur in hasce duas $\frac{\frac{a}{4}dy}{y - \frac{a}{2}} - \frac{\frac{a}{4}dy}{y + \frac{a}{2}}$, quæ ita integrentur, ut ſubtangens logiſticæ $= \frac{a}{4}$, & logarithmus $\frac{a}{4} = o$, & fiet

$$l\frac{a}{4}\cdot\frac{\sqrt{5}+1}{\sqrt{5}-1}\cdot\frac{y-\frac{a}{2}}{y+\frac{a}{2}} = l\frac{a}{4}\cdot\frac{\sqrt{5}+1}{\sqrt{5}-1}\cdot\frac{\sqrt{zz+\frac{aa}{4}}-\frac{a}{2}}{\sqrt{zz+\frac{aa}{4}}+\frac{a}{2}};$$

quæ, poſita $z = a$, fit $= l\frac{a}{4} = o$: igitur

$$AE - AD = S\frac{2dz}{z}\sqrt{zz+\frac{aa}{4}} = 2\sqrt{zz+\frac{aa}{4}} - a\sqrt{5}$$

$$+ 2l\frac{a}{4}\cdot\frac{\sqrt{5}+1}{\sqrt{5}-1}\cdot\frac{\sqrt{zz+\frac{aa}{4}}-\frac{a}{2}}{\sqrt{zz+\frac{aa}{4}}+\frac{a}{2}}.$$

Nunc me converto ad formulam $\frac{dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}}$, quæ per arcus ellypticos, & hyperbolicos integratur ex N. XVII primæ diſquiſitionis hoc modo. Poſito ſemiaxe majore (*Fig.* 17.) $GM = \frac{a\sqrt{5}}{2}$, & minore $GN = a$ deſcribatur ellypſis MN: abſcinde $GR = \frac{a\sqrt{zz-aa}}{\sqrt{zz+\frac{aa}{4}}}$, & determina arcus NQ, MQ. Tum ſecta $GL = \frac{a}{2}$, cum ſemiaxibus GL, GN deſcribe hyperbolam LO, & accipe abſciſſam $GS = \frac{az}{2\sqrt{zz-aa}}$, & determina arcum LO. Ex N. XVII diſquiſitionis primæ non curata conſtante, quam paullo infra determinabimus, habemus

$$S\frac{dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}} = \frac{z\sqrt{zz-aa}}{\sqrt{zz+\frac{aa}{4}}} + \frac{25a^2z}{16\sqrt{zz-aa}.\sqrt{zz+\frac{aa}{4}}} - \frac{5}{4}NQ - MQ - \frac{5}{4}LO, \text{ sive}$$

neglecto quadrante ellyptico, qui constans est

$$S\frac{dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}} = \frac{16z^3+9a^2z}{16\sqrt{z^2-a^2}.\sqrt{z^2+\frac{a^2}{4}}} - \frac{NQ}{4} - \frac{5}{4}LO.$$

Addenda est ejusmodi quantitas, ut, facta $z=a$, omnia evanescant. Verum, posita $z=a$, tum quantitas algebraica, tum arcus LO evadit infinitus; quare quantitas addenda exprimeretur per differentiam duarum quantitatum infinitarum. Hoc incommodum vitabis utens artificio, ut determinatis arcubus VO, VY, quorum differentia sit rectificabilis, pro arcu LO substituas arcum LY. Hanc ob rem seca

$GT = a\sqrt{\frac{1}{4}+\frac{1}{2\sqrt{5}}}$, tum $GX = \frac{\sqrt{aa+4zz}}{2\sqrt{5}}$: habebis

$\frac{z\sqrt{aa+4zz}}{2\sqrt{zz-aa}} - a.\overline{2+\sqrt{5}} = VO - VY$: Igitur

$\frac{z\sqrt{aa+4zz}}{2\sqrt{zz-aa}} - a.\overline{2+\sqrt{5}} + 2LV - LY = LO$. Peracta substitutione, factoque calculo proveniet

$$S\frac{dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}} = \frac{-z\sqrt{zz-aa}}{2\sqrt{aa+4zz}} + \frac{5a}{4}.\overline{2+\sqrt{5}} - \frac{NQ}{4}$$

$-\frac{5}{2}LV+\frac{5}{4}LY$. Ejusmodi quantitas addenda erit, ut facta $z=a$, omnia evanescant: atqui posita $z=a$, evanescit quantitas algebraica $\frac{z\sqrt{zz-aa}}{\sqrt{aa+4zz}}$, NQ fit quadrans ellypticus, & evanescit LY: igitur formula ita erit enuncianda

$$S\frac{2dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}}=AE+AD=\frac{-z\sqrt{zz-aa}}{\sqrt{aa+4zz}}+\frac{MQN-NQ}{2}+\frac{5}{2}LY=\frac{-z\sqrt{zz-aa}}{\sqrt{aa+4zz}}$$

$$+\frac{MQ}{2}+\frac{5}{2}LY.$$

Determinatis ita constantibus, adverto, hoc æquationem habere incommodi, quod facta z infinita, duplex infinitum habeatur, nempe quantitas algebraica, ac arcus LY. Quare iterum pro LY substituamus LO, qui, facta z infinita, erit nullus. Habebimus itaque

$$AE+AD=\frac{z.\overline{9a^2+16zz}}{4.\sqrt{zz-aa}.\sqrt{aa+4zz}}-\frac{5a}{2}.\overline{2+\sqrt{5}}$$

$+5LV-\frac{5}{2}LO+\frac{MQ}{2}$, in qua, si z sit infinita, sola quantitas algebraica infinita est, evanescente LO, & degenerante MQ in quadrantem ellypticum.

Quoniam tam summa, quam differentia arcuum AE, AD inventa est, liquet, dari arcus AE, AD. Verum non debeo omittere determinationem arcus AF. Hanc ob rem primam æquationem deme ex secunda, seu differentiam ex summa arcuum, & invenies

$$2AD=-\sqrt{aa+4zz}+\frac{z.\overline{9aa+16zz}}{4\sqrt{zz-aa}.\sqrt{aa+4zz}}+$$

$a\sqrt{5}$

$$a\sqrt{5} - \frac{5a}{2} \cdot \overline{2+\sqrt{5}} - 2l\frac{a}{4} \cdot \frac{\sqrt{5}+1}{\sqrt{5}-1} \cdot \frac{\sqrt{zz+\frac{aa}{4}} - \frac{a}{2}}{\sqrt{zz+\frac{aa}{4}} + \frac{a}{2}} +$$

$$5LV - \frac{5}{2}LO + \frac{MQ}{2}.$$

In hac formula licet duæ quantitates algebraicæ, facta z infinita, ambæ infinitæ sint; tamen sese destruunt, quum earum differentia non finita sit, sed infinitesima, ut mox ostendam. Reliquæ omnes finitæ sunt. Quare contractis terminis inveniemus

$$2AF = 5a - \frac{3}{2}a\sqrt{5} - 2l\frac{a}{4} \cdot \frac{\sqrt{5}+1}{\sqrt{5}-1} + 5LV + \frac{MQN}{2}.$$

Nihil jam reliquum est, nisi ut ostendam, differentiam duarum quantitatum $\sqrt{aa+4zz}$, $\frac{z \cdot 9aa+16zz}{4 \cdot \sqrt{aa+4zz} \cdot \sqrt{zz-aa}}$, facta z infinita, non finitam, sed infinitesimam esse. Quantitates duas ita dispono $\frac{aa+4zz \cdot \sqrt{zz-aa}}{\sqrt{aa+4zz} \cdot \sqrt{zz-aa}}$,

$\frac{z \cdot 9aa+16zz}{4\sqrt{aa+4zz} \cdot \sqrt{zz-aa}}$. Pono extracta radice $\sqrt{zz-aa} = z - \frac{aa}{2z}$, terminos reliquos negligo, quia in eo sunt infinitesimorum ordine, quem in calculo negligere oportet. Prima quantitas mutatur in hanc

$$\frac{aaz + 4z^3 - \frac{a^4}{2z} - 2aaz}{\sqrt{aa+4zz} \cdot \sqrt{zz-aa}} = \frac{4z^3 - a^2z + \frac{a^4}{2z}}{\sqrt{aa+4zz} \cdot \sqrt{zz-aa}};$$ atqui altera

quantitas $= \frac{4z^3 + \frac{9}{4}aaz}{\sqrt{aa+4zz} \cdot \sqrt{zz-aa}}$: ergo hae a prima dem-

 pta

pta differentia fit $= \dfrac{-\frac{13}{4}aaz + \frac{a^4}{2z}}{\sqrt{aa+4zz}\cdot\sqrt{zz-aa}}$, in qua omittenda $\frac{a^4}{2z}$ infinitesima respectu $\frac{13}{4}a^2z$: atqui $\dfrac{-13a^2z}{\sqrt{aa+4zz}\cdot\sqrt{zz-aa}}$ est infinitesima, quia in divisore z elevatur ad secundam potestatem, in numeratore tantum ad primam: differentia igitur inter duas expositas quantitates algebraicas infinitesima est.

Methodus hæc, qua ad constructionem perduxi formulam a te propositam, alias ejusdem generis formulas felicissime absolvit: quod unico dumtaxat exemplo non erit supervacaneum ostendere. Integranda sit formula $dx\sqrt{1-\dfrac{8a^2xx}{\overline{aa+xx}^2}}$. Eadem utor substitutione $\frac{aa+xx}{x}=2z$, quæ formulam propositam in hanc convertit $\frac{dx}{z}\sqrt{zz-2aa}$. Ex substitutione item deducimus $x=z\pm\sqrt{zz-aa}$, quæ docet, duplicem x eidem z respondere. Eliminans demum a formula elementum dx invenio $\frac{dz}{z}\sqrt{zz-2aa}\pm\dfrac{dz\sqrt{zz-2aa}}{\sqrt{zz-aa}}$.

Ut claritati consulamus, atque facilitati, formulas revocemus ad figuras geometricas. Primo describe hyperbolam, quæ sit locus hujus æquationis $aa+xx=2zx$. Describitur autem hoc modo. Constituantur ad angulum rectum (*Fig.* 18.) CO, CQ, in hac accipe CA $=a$, cui sit æqualis, & perpendicularis AB, quam divide bifariam in D, tum junge, & produc CD. Inter assymptotos CO, CD describe hyperbolam transeuntem per punctum B; hæc erit locus quæsitus, & existentibus CP,

CQ

$CQ = x$, PM, $QN = z$. Constructio ista satis docet, eidem z duplicem x respondere. Nam abscinde $AV = z$. Per V duc MN parallelam CQ, & demitte normales MP, NQ, erunt CP, CQ duæ x eidem z respondentes: ergo $CQ = z + \sqrt{zz - aa}$, & $CP = z - \sqrt{zz - aa}$.

Nunc aliam curvam delineemus, cujus abscissæ $= x$, ordinatæ $= \frac{a\sqrt{zz - 2aa}}{z}$. Quoniam istæ, ut patet, sunt imaginariæ, nisi sit $z > a\sqrt{2}$, abscinde $AE = a\sqrt{2}$, age per E ordinatam FG, & demitte normales FH, GK; in punctis H, K duo illi, quibus curva constat, rami incipient, quorum primus erit HL existente $CL = a$, alter erit KT assymptoticus rectæ LT parallelæ CQ.

Ductis infinite proximis MN, mn, item MPR, mpr, & NQS, nqs, manifestum est, elementum

$QSsq = \frac{adx}{z}\sqrt{zz - 2aa}$, quia crescente spatio KSQ, crescit etiam abscissa x; sed elementum $PRrp =$

$-\frac{adx}{z}\sqrt{zz - 2aa}$, quia crescente spatio HRP, abscissa x minuitur. Quare dx opportune eliminata habebimus

$$PRrp = -\frac{adz}{z}\sqrt{zz - 2aa} + \frac{adz\sqrt{zz - 2aa}}{\sqrt{zz - aa}}$$

$$QSsq = \frac{adz}{z}\sqrt{zz - 2aa} + \frac{adz\sqrt{zz - 2aa}}{\sqrt{zz - aa}}\text{: igitur}$$

$$QSsq - PRrp = \frac{2adz}{z}\sqrt{zz - 2aa}$$

$$QSsq + PRrp = \frac{2adz\sqrt{zz - 2aa}}{\sqrt{zz - aa}}\text{, \& facta integratione}$$

$$KQS - HRP = S\,\frac{2adz}{z}\sqrt{zz - 2aa}$$

KQS

$KQS + HRP = S \dfrac{2 a d z \sqrt{zz - 2aa}}{\sqrt{zz - aa}}$, quæ summatoriæ ita accipiendæ sunt, ut evanescant facta $z = a\sqrt{2}$.

Ut integrem formulam $\dfrac{dz}{z}\sqrt{zz - 2aa}$, eam in hunc modum dispono $\dfrac{z dz}{\sqrt{zz - 2aa}} - \dfrac{2aadz}{z\sqrt{zz - 2aa}}$. Ex his prima integrabilis est, & ejus summa $= \sqrt{zz - 2aa}$; altera dat arcum circularem, cujus radius $= a\sqrt{2}$, secans $= z$. Quare facillima est, constructio (*Fig.* 19.). Radio a e $= a\sqrt{2}$ describatur quadrans circularis e d b, & ducta tangente e c, applicetur secans a c $= z$, erit tangens e c $= \sqrt{zz - 2aa}$, arcus e d $= S \dfrac{2aadz}{z\sqrt{zz - 2aa}}$: ergo

$KQS - HPR = 2a . \sqrt{zz - 2aa} - ed = 2a . \overline{ec - ed}$.

Altera formula $\dfrac{dz\sqrt{zz - 2aa}}{\sqrt{zz - aa}}$ construitur sola ellypsi rectificata, ut constat ex N. VII primæ disquisitionis. Sumpta a f $= a$ describatur ellypsis f g b, & in axe majore sumatur a i $= \dfrac{a\sqrt{zz - 2aa}}{\sqrt{zz - aa}} . \sqrt{2}$, & ducta ordinata i g determinetur arcus f g: qui arcus facilius determinatur, si tangens f k, quæ ducitur, cuplicetur, ut sit k h $=$ f g, & jungatur a h: erit $S \dfrac{dz\sqrt{zz - 2aa}}{\sqrt{zz - aa}} = \dfrac{z\sqrt{zz - 2aa}}{\sqrt{zz - aa}} -$ f g: ergo

$KQS + HPR = 2a . \dfrac{z\sqrt{zz - 2aa}}{\sqrt{zz - aa}} - fg$.

Quoniam tam summa, quam differentia spatiorum KQS, HRP inventa est, constat eadem spatia pariter inveniri. Verum ad aliquot determinationes faciendas utrumque inveniamus.

KQS

$$KQS = a . \overline{\sqrt{zz - 2aa} + \frac{z\sqrt{zz - 2aa}}{\sqrt{zz - aa}} - ed - fg}$$

$$HPR = a . \overline{\frac{z\sqrt{zz - 2aa}}{\sqrt{zz - aa}} - \sqrt{zz - 2aa} + ed - fg}.$$

Determinemus spatium HLC, quod obtinetur ex secunda formula posita z infinita. Quantitates algebraicæ duæ sese mutuo destruunt, quia earum differentia fit infinitesima, quod facile probatu est: nam ita illas exponemus

$\frac{\sqrt{zz - 2aa}}{\sqrt{zz - aa}} . \overline{z - \sqrt{zz - aa}}$: atqui $\sqrt{zz - aa} = z - \frac{aa}{2z}$ omissis subsequentibus terminis, qui certe ob exiguitatem negligendi sunt: ergo habemus $\frac{\sqrt{zz - 2aa}}{\sqrt{zz - aa}} . \frac{aa}{2z}$, quæ quantitas infinitesima est, ut cuique patet: ergo $HLC = a . \overline{edb - fgb}$.

Nunc producta GK in X, requiramus spatium KXT clausum inter curvam, & assymptotum. Quoniam $CK = a . \overline{\sqrt{2} + 1}$, & $CQ = z + \sqrt{zz - aa}$, erit $KQ = \overline{z + \sqrt{zz - aa} - a\sqrt{2} - a}$: ergo rectangulum $XKQ = a . \overline{z + \sqrt{zz - aa} - a\sqrt{2} - a}$, a quo auferatur spatium KQS, & remanebit spatium

$$XKS = a . \overline{z + \sqrt{zz - aa} - \sqrt{zz - 2aa} - \frac{z\sqrt{zz - 2aa}}{\sqrt{zz - aa}} - a\sqrt{2} - a + ed + fg}.$$

Si ponamus z infinitam, quantitates algebraicæ, ubi z ingreditur, evanescunt, quia earum differentia est infinitesima, ut facile methodo, qua paullo ante usi sumus, probare potes: ergo spatium infinite longum

$KXT = a . \overline{edb - a\sqrt{2} + fgb - a}$, sive

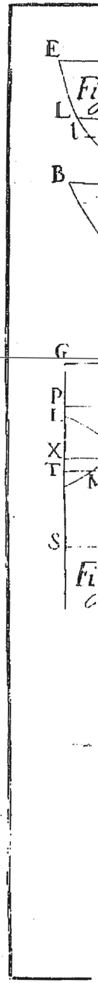
$KXT = a . \overline{edb - ae + fgb - af}$.

Duo exempla, quæ attulimus, certissimum te faciunt, Vir Clarissime, formulam

$dx\sqrt{1+\frac{ma^2x^2}{(pa^2+qx^2)^2}}$, quicumque ſint valores m, p, q vel poſitivi, vel negativi, ſemper eadem methodo ad conſtructionem perduci. Hæc autem formula hoc proprium habet, quod ſimul & rectificationem ſectionum conicarum, & quadraturam requirit, quum formulæ ferme omnes, quæ in ſecunda diſquiſitione tractatæ ſunt, ſola ſint rectificatione contentæ. Vale

Ex Collegio Sanctæ Luciæ octavo Kal. Februarii 1758.

OPU-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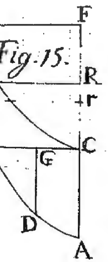

Fig.15.

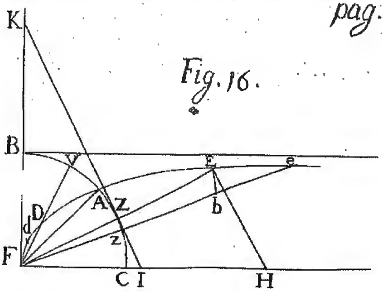

Fig.16.

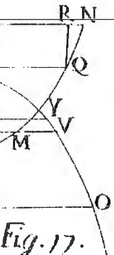

Fig.17.

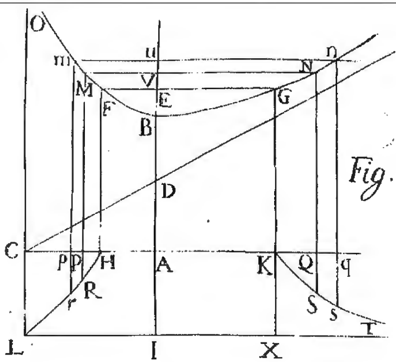

Fig.18.

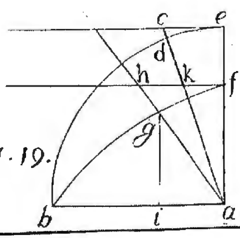

Fig.19.

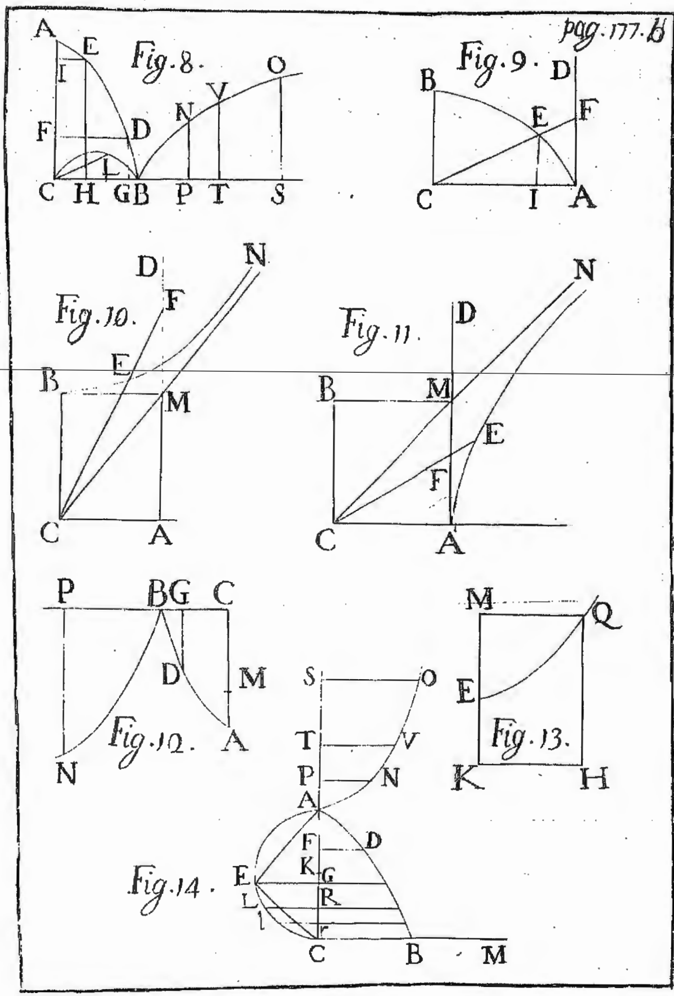
Fig. 8.
A E I F D L C H G B P T S N Y O
Fig. 9.
B D E F C I A
Fig. 10.
D F N E B M C A
Fig. 11.
D N B M E F C A
Fig. 12.
P B G C D M N A
Fig. 13.
M Q E K H
Fig. 14.
S O T V P N A F D E K G L R l r C B M

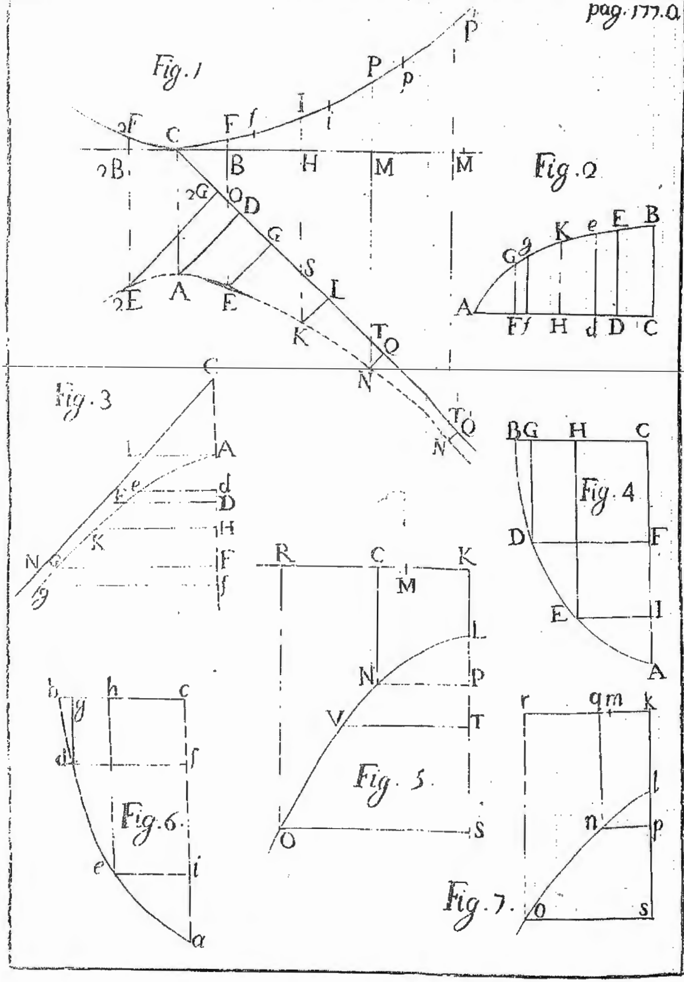
Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6
Fig. 7

OPUSCULUM TERTIUM.

EPISTOLA

In qua ad examen vocatur argumentum, quo Galileus refellit hypothesim gravium ea lege descendentium, ut velocitates sint spatiis peractis proportionales.

VINCENTIUS RICCATUS

JOANNI BAPTISTÆ NICOLAO

In Seminario Tarvisino Mathematicæ, & Phylosophiæ Professori

S. P. D.

NIhil mireris, Vir Clarissime, quod, postquam ad accuratum examen revocasti argumentum illud, quo Celeberrimus Galileus confutat hypothesim statuentem, mobile actum a gravitate constante acquirere velocitates spatiis proportionales, in quamplurimas dubitationes incideris, & incertus hæreas, quodnam judicium de illo ferendum sit: namque acutissimi geometræ, qui non minus anteacto, quam præsenti sæculo floruerunt, hoc idem experti sunt ambigentes, utrum argumentum illud sit necne recensendum inter demonstrationes geometricas. Quapropter, ut ex litteris a Cazreo ad Gasendum datis colligere licet, neglecta prorsus Galilei ratione, hypothesis refutata defensores non paucos nacta est: neque adhuc fortasse in exilium a physica expulsa esset, nisi Petrus Fermatius Geometra in primis acutus methodo firmissima more veterum concinnata eamdem ad patens absurdum deduxisset.

Neque prorsus iniquæ hujuscemodi dubitationes putandæ sunt, quia Galileus (fateri fas est) præter morem subobscurius loquutus est. Ut petitionibus tuis faciam satis, navabo operam, ut omni obscuritate sublata Galilei ratiocinium exponam dilucide, ejusque vim penitus patefaciam. Tuum erit deinceps de eodem judicium ferre.

Antequam recipio, quod spopondi, per te mihi liceat, verba ipsa Galilei ex italico idiomate in latinum translata describere, ut certior fieri possis, a me non novum confici, sed vitus Galilei argumentum clarius exponi. Ita Galileus Salviatum suum loquentem inducit. *Quum velocitates eamdem habent rationem, quam spatia transacta, aut transigenda, hujusmodi spatia æqualibus conficiuntur temporibus. Itaque si velocitates, quibus mobile cadens confecit spatium quatuor ulnarum, duplæ fuerunt velocitatum, quibus iter habuit per duas primas ulnas (quemadmodum spatium spatii duplum est), igitur horum itinerum tempora æqualia sunt. Atqui idem mobile percurrere non potest & quatuor, & duas ulnas eodem tempore præter quam in motu instantaneo. Verum videmus grave decidens motum suum finito tempore persolvere, & citius duas ulnas, quam quatuor percurrere. Falsum igitur est, velocitatem crescere quemadmodum spatium.*

Verba Galilei accepisti: accipe nunc, quomodo ratio hæc per me clarissime exponatur. Mobilia duo æqualia A, B descendant (*Fig.* 1.) motu accelerato per spatia AC, BD, quæ sint, caussa exempli, ut 2 : 1. Quoniam hypothesis poscit, ut mobilia acquirant velocitates spatiis proportionales, velocitates in C, D erunt ut 2 : 1. Dividatur spatium BD in numerum infinitum infinitesimorum spatiorum æqualium, ut Dn, n2n, 2n3n &ce. Similiter divide spatium AC in æqualem numerum minimorum spatiorum, ut Cm, m2m, 2m3m & ce. Liquet, quodlibet ex his spatiolis esse ad quodlibet ex illis, in quæ divisum est spatium BD, ut 2 : 1.

Quoniam AC : BD :: 2 : 1, & Cm : Dn :: 2 : 1 : erit Am : Bn :: 2 : 1 : Igitur velocitates in m, n sunt ut 2 : 1. Similiter quando non minus Am : Bn, quam m2m : n2n est ut 2 : 1, velocitates in 2m, 2n erunt ut 2 : 1. Ita progrediens demonstrabo, spatia omnia analoga, & velocitates in omnibus punctis analogis sese habere ut 2 : 1.

Docent leges motus æquabilis, tempora æqualia esse, quum velocitates spatiorum confectorum servant proportionem. Quare quum spatiola mC, nD transiguntur velocitatibus, quæ sunt ut spatia, hoc est, ut 2 : 1, iisdem prorsus temporibus conficientur. Eadem ratione æqualia probabo tempora, quibus percurruntur spatiola 2mm, 2nn, quia tam ipsa, quam velocita-

tes

tes tenent rationem 2 : 1. Sic diſcurſum protrahens ab elemento ad elementum probabo omnia elementa analoga ſpatiorum A C, B D æqualibus temporibus confici: igitur etiam integra ſpatia A C, B D eodem prorſus tempore percurrentur.

Verum evidens eſt, corpus B itinerari per ſpatium B D eo ipſo tempore, quo A deſcendit per primum dimidium ſpatii A C, quod æquale eſt B D: igitur corpus A conficit tum integrum ſpatium A C, tum primum ejus dimidium eodem tempore. Quod conſectarium illud ipſum eſt, quod antea collegerat Galileus.

Munere meo functus ſum ego, tu tuum exequere, & tribunal aſcendens fac ſententiam feras, utrum ratio expoſita locum habeat inter demonſtrationes præditas evidentia geometrica. Ne tamen præceps ſis in judicio ferendo, narrem ſinas, quid mihimetipſi accederit, quum ſtudens, ac meditans judicis partes aſſumpſi. Statim ac Galilei cogitatum tam bono in lumine collocavi: niſi hæc demonſtratio eſt, mecum ipſe inquiebam, opus erit, ſexcentas reſpuere, quæ eadem methodo procedunt, neque vim majorem præſeferunt. Attamen nemo unus eſt inter geometras, qui eaſdem libentiſſime non recipiat. Quod ſi homines doctiſſimi de Galileana ratione diu multumque dubitarunt, huic cauſſæ unice tribuendum videtur, quod propter obſcuritatem, qua obducta erat, ejus vim, ut par erat, minime perceperunt. Æquam adeo certamque ſententiam hanc meam arbitrabar, ut in litteris Corticellio datis, & editis tomo primo Opuſ. decimo, quibus Joanni Babtiſtæ Baliano gloriam, qua privatus erat, reſtituo, luculenter ſcriperim, abſurdum a Fermatio propoſitum re vera nihil differre ab eo, quod protulit Galileus, atque probavit, tametſi Farmatius uſus fuerit clariore, atque exactiore demonſtratione.

Verum repetenti mihi non ita pridem eamdem Galilei probationem iterum excitatæ ſunt veteres dubitationes, quæ non ex eo oriuntur, quod Auctoris mentem non plane aſſequutus fuerim, ſed immo quod aſſequutus fuerim clarius, atque penitius. Eas paullatim evolvam: tu, poſtquam audieris, quanti ducendæ ſint, judicabis. Libenter tibi ultro concedo ſpatiola omnia m C, 2 m m & ce. percurri eo tempore, quo percurruntur ſpatiola analoga n D, 2 n n & ce., dummodo excipias prima A i m, B i n. Hiſce primis aptari non poteſt ratio, quæ valet in aliis omnibus. Etenim

nim quum in punctis m, n velocitates finitæ sint, & earum incrementa per minimas m C, n D infinitesima, licet, ut cuique notum est, supponere, corpora per elementa m C, n D æquabili motu cieri: at in primis elementis A i m, B i n velocitates in punctis A, B nullæ sunt, in punctis i m, i n sunt aliquæ quidem certe: ergo quum mobilia per spatiola A i m, B i n motu æquabili iter non habeant, nullo modo probari potest, horum itinerum æqualia esse tempora. Si vero, inquam, differentia inter hæc tempora, quibus prima spatiola peraguntur, finita esset, quomodo ex æqualitate temporum, quibus reliqua spatiola analoga percurruntur, quomodo colligeretur, æqualia esse tempora, quibus percurruntur finita spatia A C, B D?

~~Video, ex hax dubitatione aliam tibi protinus suboriri. Si~~ res ita sese habet, tecum ipse ais, æqualitas temporum probabitur quidem in illis spatiolis, quorum finita est distantia a punctis A, B, sed non in illis, quæ distant per infinitesimum; quia in his augmenta velocitatum habent rationem finitam cum velocitatibus primitivis, atque adeo non licet considerare motum tamquam uniformem. Attamen in his quoque, dummodo prima non sint, videor, tibi æqualitatem temporum non negare. Non nego, Vir Clarissime, immo ultro tibi hanc æqualitatem concedo, neque paralogismi periculum pertimesco: dummodo spatiola prima non sint, quemadmodum non sunt elementa $i\,m\,\overline{i-1}\,m$, $i\,n\,\overline{i-1}\,n$. Etenim quamquam incrementa velocitatum sunt in ratione data ad velocitates, quibus prædita erant mobilia in punctis i m, i n, neque propterea licet considerare tamquam æquabilem motum per spatiola integra $i\,m\,\overline{i-1}\,m$, $i\,n\,\overline{i-1}\,n$: tamen quis prohibet dividere hæc duo elementa in æqualem numerum aliorum elementorum infinitesimorum secundi gradus, in quibus, quum velocitatum augmenta minimam rationem habeant ad velocitates primitivas, motus spectari potest tamquam æquabilis. Re autem sic se habente proclive est demonstratu, mobilia eodem tempore transigere spatiola analoga infinitesima secundi ordinis, atque adeo spatiola infinitesima primi ordinis $i\,m\,\overline{i-1}\,.\,m$, $i\,n\,\overline{i-1}\,.\,n$.

Verum quanam de caussa, audire videor te mihi objicientem, quanam de caussa non licet uti eadem methodo in spatiotiolis A i m, B i n, eademque dividere in spatia minima secundi ordinis. Licet, neque enim ulla lege vetitum est. At hujusmodi divisio nulli tibi erit utilitati: nam in omnibus analogis spatiolis æqualitatem temporum demonstrabis, sed non in primis. Quod si, repetere fas est, finita est differentia temporum, quibus percurruntur prima duo spatiola analoga, ad quemcumque tandem ordinem infinitesimorum pertineant, recipiendumne erit consectarium statuens, tempora per A i m, B i n æqualia esse?

Sed suppetias tibi feram, & omni te angustia liberabo, demonstrans, jure bono vocari posse æqualia tempora illa, quibus integræ lineæ A C, B D conficiuntur, tametsi differentia temporum, quibus prima elementa conficiuntur, finita sit. Quomodo, ais, componi hæc possunt? Quomodo? quia tempus, quo conficitur A C, idem dic de B D, infinitum est. Neque hoc supponere placet, sed geometrice demonstrare. Adesto animo. Hypothesim hanc mihi confingo, quam, ut brevius loquar, fictam deinceps appellabo. Pono mobile velocitate acquisita in puncto i m percurrere primum spatiolum A i m; ea vero quam aquisivit in puncto $\overline{i-1}\,m$ percurrere spatiolum alterum i m $\overline{i-1}\,n$, atque ita deinceps. Evidens est in omnibus punctis spatiolorum, extremis exceptis, majorem esse mobilis velocitatem in hypothesi ficta, quam in hypothesi vera motus continenter accelerati: igitur in hypothesi ficta breviori tempore descendet per lineam A C, quam in vera naturæ hypothesi: atqui in ficta hypothesi conficitur A C tempore infinito: ergo etiam in vera. Argumentum, quod est a fortiori, nullam recipit responsionem.

Infinitum vero esse tempus in hypothesi ficta ita demonstro. Voco $= dx$ quodlibet ex spatiolis, ut A i m. Liquet velocitates in punctis i m, $\overline{i-1}.m$, $\overline{i-2}\,m$ & ce. . . . . C exprimi per hanc seriem dx, $2dx$, $3dx$ & ce. . . . . . . $\overline{i+1}\,dx$, in qua i est numerus infinitus, quia velocitates sunt ut spatia peracta. Tempora vero, quum sint ut spatiola divisa per velocitates, in successivis spatiolis æqualibus exprimentur per terminos seriei

$\frac{dx}{dx}$, $\frac{dx}{2dx}$, $\frac{dx}{3dx}$ & ce.......... $\frac{dx}{i+1.dx}$, sive

1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ & ce.............. $\frac{1}{i+1}$. Quapropter tempus, quo in ficta hypotesi conficitur tota A C, exprimetur per summam infinitorum terminorum hujus seriei decrescentis, quæ est notissima series harmonica. Geometrarum nemini in præsens ignotum est, seriei harmonicæ summam esse infinitam: igitur infinitum est tempus, quo in ficta hypothesi mobile a puncto A pervenit ad C: ergo a fortiori etiam in hypothesi vera. Idem dicas velim de tempore, quo mobile B pervenit ad D. En tibi maxime simplicem demonstrationem ejus absurdi, quod primum detectum est, atque demonstratum a Fermatio in celeberrima epistola ad Gassendum.

His positis si differentia temporum, quibus percurruntur spatiola prima A i m, B i n, aut quibus percurruntur spatia integra A C, B D, finita est, hujusmodi differentia ad ipsa tempora, quæ demonstrata sunt infinita, minorem rationem habent quacumque data: igitur tempora, quibus conficiuntur lineæ A C, B D, in aliquo vero sensu vocari possunt æqualia. Verum putasne, hujusmodi æqualitatem quidquam facere ad sustinendum Galilei argumentum? Nihil sane. Etenim nullo modo hoc consectarium descendit: ergo motus est instantaneus. Nam rectæ A C, B D, aut recta A C, ejusque dimidium non conficiuntur temporibus ita æqualibus, ut eorum differentia nulla sit; sed inter ipsa finita intercedit differentia, qua mobile descendit per secundum dimidium lineæ A C, hoc est percurritur linea finita tempore finito. Nihil itaque est, cur timeamus motum instantaneum, quem minatus fuerat Galileus.

Sed subiratum te conspicio, atque ita mihi opponentem. Recte quidem, si differentia temporum per prima infinitesima elementa finita sit. At quis certo asserit finitam? hactenus supposita est, non probata. Rogo te, ut placida mente advertas, nihil aliud mihimetipsi proposuisse me, nisi ut in dubium revocarem Galilei argumentum: quem ob finem satis mihi erat probare, illud invalidum esse, imo paralogisticum, si differentia inter prædicta tempora foret finita. Ad Galileum, aut ad alios, qui tue-

ri

ri ejus partes velint, pertinet probare, differentiam temporum per prima spatiola duo non finitam esse, sed infinitesimam. Neque ad hoc sufficit dicere, infinitesima esse prima spatiola; quia licet ipsa infinitesima sint, tempora tamen, & eorum diferentiæ finitæ esse possunt. Si vero aliquis conetur demonstrare prædictam temporum differentiam finitam non esse, ego tibi polliceor, eum oleum atque operam perditurum.

Sed quid erit pretii, si omni te angustia, atque dubitatione liberavero? Ajo itaque audacter, finitam esse differentiam temporum, quibus duo prima spatiola conficiuntur: atque hujus veritatis habeo geometricam demonstrationem, quam postremo placet ad te scribere. Notissimum tibi est, elementa temporum esse in ratione directa spatiolorum, & inversa velocitatum. Igitur acceptis velocitatum reciprocis, elementa temporum erunt tum ut elementa spatiorum, tum ut velocitatum reciprocæ.

Sancito hoc theoremate, quod ad unum omnes recipiunt, venio ad hypothesim corporum ea lege descendentium, ut velocitates sint, quemadmodum spatia. Quare si A S exprimunt spatia, S V exprimentes (*Fig.* 2.) velocitates erunt ordinatæ ad lineam rectam A V. Igitur curva, cujus ordinatæ sunt velocitatibus reciprocæ, erit hyperbola apolloniana inter assimptotos O G R. Accepto minimo spatio S s, tempus per S s erit in ratione composita tum S s, tum R S, sive in ratione simplici rectanguli R S s, aut minimæ areæ R S s r: ergo integrando tempus per A S erit ut area hyperbolica O A S R, quæ area quum infinita sit, ut omnibus notum est, iterum probatum remanet, tempus per finitam A S infinitum esse.

Ad rem nostram accedo propius. Accipe minimum spatium A M, atque hoc spectet ad quemcumque volueris ordinem infinitesimorum, & duc ordinatam M G, habebimus tempus per A M expressum ab area O A M G. Similiter secta A M æqualiter in N, ductaque ordinata N F, tempus per A N repræsentatur ab area O A N F: igitur differentia temporum per A M, A N exprimitur per aream F N M G: atqui hæc finita est: ergo temporum differentia per A M, A N finita est.

Ut finita probetur area F N M G, ducantur per puncta F, G parallelæ abscissis rectæ K F L, G I H, & vocetur $= aa$ rectan-

rectangulum constans ex abscissis, & ordinatis hyperbolicis, quod certe finitum est. Area F N M G est major rectangulo G I N M, quod est dimidium rectanguli H A M, sive aa: ergo area F N M G $> \frac{1}{2} aa$. Eadem area est minor rectangulo F N M K, aut F L A N, aut aa: ergo F N M G $< aa$. Quamobrem area F N M G, quæ media est inter duas quantitates finitas, major una, minor altera, sine dubio finita sit, necesse est. Q: E: D.

Ex hac demonstratione colligas velim, eamdem semper esse differentiam temporum, quibus percurruntur duo spatia quæcumque, quæ sint in ratione dupla. Accipe A R, quæ sit dimidium A S. Differentia temporum, quibus conficiuntur A P, A S, est area Q P S R. Atqui quum sit A N: A M:: A P: A S constat, areas F N M G, Q P S R æquales esse: ergo & ce. Hinc alia habetur demonstratio areæ F N M G finitæ, licet infinitesimæ sint A N, A M: quia quum finitæ sint A P, A S, area Q P S R finita est quidem certe. Horum spatiorum hyperbolicorum valorem proximum vero, quantum volueris, per series possem exhibere. Sed abuterer tua patientia, si in rebus hisce notissimis diutius immorarer.

His attente consideratis quid tibi videtur? Num reprehendendi sunt qui dubitant, utrum Galilei argumentum inter demonstrationes geometricas sit connumerandum? Sententiam, arbitror, a te pronunciatam esse adversus Galilei argumentum, quod non dubii paralogismi condemnas. Adverte, Vir Clarissime, quanta cautione opus sit, dum transitus ab infinitesimis ad finita faciendus est. Primo intuitu unusquisque jurasset, demonstrativum esse discursum, quo Galilei rationem illustravi. Attamen in eo latebat fallacia, quam gaudeo me detexisse, quia tibi morem gerere, tuisque petitionibus satisfacere potui. Vale

Bononiæ Nono Kal. Decembris 1757. Tarvisium ad Clariss. Joannem Baptistam Nicolaum.

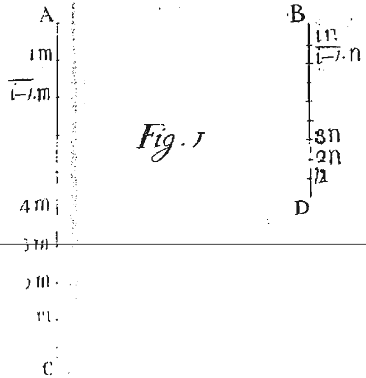
A
1 m
$\overline{i-1}$. m
4 m
3 m
2 m
m
C
B
1 n
$\overline{i-1}$. n
3 n
2 n
n
D

Fig. 1

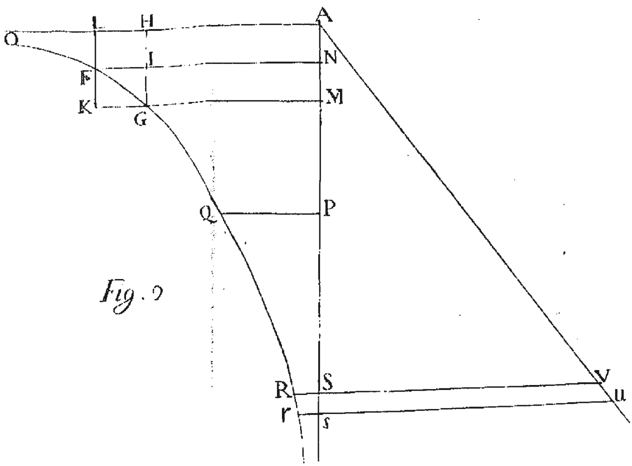
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Fig. 2

OPUSCULUM QUARTUM.

EPISTOLA

In qua exhibetur formula generalis æquationum, quæ radicem habent cardanicæ similem, atque ejus ope formulæ aliquot in trinomia realia resolvuntur, & cottesianum theorema demonstratur.

VINCENTIUS RICCATUS

FRANCISCO BENALEÆ (*a*).

S. P. D.

PErgratum mihi accidit, quod ea tibi, Vir Clarissime, probata fuerint, quæ scripsi in quarto opusculo tomi primi, ubi ago de æquationibus recipientibus radicem similem radici cardanicæ. Approbatio tua effecit, ut easdem æquationes mente repetens nonnulla animadverterim, quæ & earum naturam patefaciunt, & usum amplificant. Hæc in præsens ad te mitto: tu fac judices, utrum aliquid pretii sint habitura.

In opusculo quarto Tomi primi pag. 52. tabulam invenies, cu-

A a

(a) *Antequam has litteras acciperet Franciscus Benalea, miserandum in modum Ulisipone diem obiit supremum. Natus erat honesto loco Tarvisii, ibique physicas, & mathematicas facultates didicerat a Jordano Comite Riccato Fratre meo, cui erat carissimus. Quum ingenio Franciscus polleret acerimo, neglecto prorsus jure, cui antea operam dederat, ad hæc studia animum unice intendit, in iisque adeo profecit, ut nihil non esset ab eo, si diutius vixisset, expectandum. Sed ubi statum ecclesiasticum amplexus est, ac sacerdotio initiatus, ita ad studia theologica se contulit, ut geometrica ne desereret. Socius accitus ab Angelo Emo Equite Ornatissimo, qui navi venetæ præest, plura cum eo maritima itinera obivit. Sed dum a Corcyra navigat Ulisiponem, in Oceano, exorta fœdissima tempestate, amisso gubernaculo, in summa commeatus, ac præsertim aquæ caritate ita debilitatus est, ut statim ac portum attigit, lætali febre correptus, plenus ea, quam semper coluerat, pietate ac religione vitam finierit.*

cujus auxilio æquationes illæ usque ad gradum decimum quartum efformantur: præterea methodum facilem docui, qua tabula illa ad quoscumque gradus extenditur. Verum formulam generalem non exhibui, in qua omnes hujus generis æquationes continerentur. Hoc præstabo primum; quando præstare facile possum, utens methodo inveniendi terminos generales serierum algebraicarum, quam tradidi Capite secundo Commentarii *De Seriebus recipientibus summam algebraicam, aut exponentialem*.

Fac tibi ob oculos ponas tabulam, cujus paullo ante mentionem feci. Series prima verticalis, quæ subest termino mn, est series arithmetica, cujus scilicet differentiæ primæ constantes sunt. Ejus terminus generalis statim cognoscitur $= p$, denotante p gradum æquationis. Altera series, quæ subest termino $m^2 n^2$, est algebraica secundi ordinis, quæ habet constantes differentias secundas. Ejus terminus generalis, ut constat ex commentario de Seriebus, hac formula continetur $A + Bp + Cpp$, quæ debet $= 2$ posita $p = 4$, debet $= 5$ posita $p = 5$, debet $= 9$ facta $p = 6$. Igitur habebimus tres æquationes

$$A + 4B + 16C = 2$$
$$A + 5B + 25C = 5$$
$$A + 6B + 36C = 9$$

Deme primam ex secunda, secundam ex tertia, & duas æquationes invenies

$$B + 9C = 3$$
$$B + 11C = 4$$

Dematur item ex altera prima, & fiet

$2C = 1$, sive $C = \frac{1}{2}$. Quo valore in aliis æquationibus opportune substituto, nascetur $B = -\frac{3}{2}$, $A = o$. Igitur terminus generalis secundæ seriei substantis termino $m^2 n^2$ fiet

$$= \frac{-3p + pp}{2} = \frac{p \cdot \overline{p - 3}}{2}.$$

Similiter series tertia, cui superstat terminus $m^3 n^3$, est algebraica tertii ordinis, & habet tertias differentias constantes. Ejus terminus generalis hac formula includitur

A

$A + Bp + Cpp + Dp^3$, quæ debet æquare 2, 7, 16, 30 facta successive $p = 6, 7, 8, 9$. Quatuor ergo nascuntur æquationes

$$\begin{aligned} A + 6B + 36C + 216C &= 2 \\ A + 7B + 49C + 343D &= 7 \\ A + 8B + 64C + 512D &= 16 \\ A + 9B + 81C + 729D &= 30 \end{aligned}$$

Singulæ æquationes istæ a sequentibus detrahantur, & tres orientur æquationes

$$\begin{aligned} B + 13C + 127D &= 5 \\ B + 15C + 169D &= 9 \\ B + 17C + 217D &= 14 \end{aligned}$$

Facta ut antea singularum deductione duæ sequentes orientur

$$\begin{aligned} 2C + 42D &= 4 \\ 2C + 48D &= 5 \end{aligned}$$

& prima ab altera deducta fiet

$6D = 1$, sive $D = \frac{1}{6}$. Demum opportunis peractis substitutionibus $C = \frac{-9}{6}$, $B = \frac{20}{6}$, $A = 0$. Quapropter seriei terminus generalis erit $\frac{20p - 9pp + p^3}{6} = \frac{p \cdot \overline{p-4} \cdot \overline{p-5}}{2 \cdot 3}$.

Simili utens methodo in reliquis seriebus, quæ sunt omnes algebraicæ, quarum gradus unitate crescit, invenies terminos generales ordinatim esse

$$\frac{p \cdot \overline{p-5} \cdot \overline{p-6} \cdot \overline{p-7}}{2 \cdot 3 \cdot 4}$$

$$\frac{p \cdot \overline{p-6} \cdot \overline{p-7} \cdot \overline{p-8} \cdot \overline{p-9}}{2 \cdot 3 \cdot 4 \cdot 5}$$

$$\frac{p \cdot \overline{p-7} \cdot \overline{p-8} \cdot \overline{p-9} \cdot \overline{p-10} \cdot \overline{p-11}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

$$\frac{p \cdot \overline{p-8} \cdot \overline{p-9} \cdot \overline{p-10} \cdot \overline{p-11} \cdot \overline{p-12} \cdot \overline{p-13}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot}$$; atque ita deinceps.

Quæ quum ita ſint æquatio generalis, cui eſt radix ſimilis cardanicæ, invenitur eſſe

$$x^{p} - p\,a\,x^{p-2} + \frac{p.\overline{p-3}}{2}\,a^{2}x^{p-4} - \frac{p.\overline{p-4}.\overline{p-5}}{2.3.}\,a^{3}x^{p-6}$$

$$+ \frac{p.\overline{p-5}.\overline{p-6}.\overline{p-7}}{2.3.4}\,a^{4}x^{p-8} - \frac{p.\overline{p-6}.\overline{p-7}.\overline{p-8}.\overline{p-9}}{2.3.4.5}\,a^{5}x^{p-10}$$

$$+ \frac{p.\overline{p-7}.\overline{p-8}.\overline{p-9}.\overline{p-10}.\overline{p-11}}{2.3.4.5.6.}\,a^{6}x^{p-12} \ldots\ldots - b = 0.$$

In hac formula animadverte, omittendum eſſe terminum illum, in quo p incipit eſſe minor eo numero, qui deducendus eſt, & terminos omnes conſequentes.

Detecta generali formula non injucundum tibi erit, Vir Clariſſime, inſpicere, qua facilitate, & elegantia binomia aliquot & trinomia in factores reales ſecundi gradus reſolvantur, & Cotteſii doctiſſimi viri theorema vulgatiſſimum demonſtretur. Namque his dumtaxat exemplis, quæ jam nota ſunt, uſum æquationis noſtræ juvat patefacere. Hanc ob rem memento, me in ejuſdem opuſculi parte altera docuiſſe, radicem æquationis, poſita a poſitiva, ſemper ita exprimi $x = 2.\mathrm{C}.\frac{\varphi}{p}$; exiſtente φ arcu, vel logarithmo, cujus ſinus totus $= a^{\frac{1}{2}}$, & coſinus $= \frac{b}{2.a^{\frac{p-1}{2}}}$, & ſinus poſitivus eſt. Si $\frac{bb}{4} > a^{p}$, accipiendi ſunt coſinus hyperbolici; quo in caſu exiſtente p impari, $\mathrm{C}b.\frac{\varphi}{p}$ habet unicum tantum valorem realem, exiſtente p pari habet duos, poſitivum unum, alium negativum. Contra ſi $\frac{bb}{4} < a^{p}$, capiendi erunt coſinus circulares, in quo caſu $\mathrm{C}c.\frac{\varphi}{p}$ tot valores habet, quot p continet unitates. Qui valores, poſito φ arcu circumferentia minore, & circumferentia $= c$, erunt

$$Cc.\frac{\varphi}{p},\ Cc.\frac{c+\varphi}{p},\ Cc.\frac{2c+\varphi}{p}\ldots\ldots Cc.\frac{\overline{p-1}.c+\varphi}{p}.$$

Relicto casu primo, qui nos deducit ad cosinus hyperbolicos, specto dumtaxat alterum, in quo, quum ad cosinus circulares ducamur, æquatio radices omnes reales continet. Assumo trinomium $zz-xz+a=o$. Si in inventa generali æquatione substituam pro x ejus valorem a trinomio exhibitum, nempe $z+\frac{a}{z}$ manifestum est, novam oriri formulam, quæ liberata a divisoribus erit resolubilis in tot trinomia similis formulæ, quot sunt valores x. Facta substitutione nascitur formula $z^p+\frac{a^p}{z^p}-b=o$, sive trinomium $z^{2p}-bz^p+a^p=o$. Quoniam existente $\frac{bb}{4}<a^p$, x habet omnes valores reales, quos antea invenimus, constat, trinomium, cujus gradus est $2p$, resolvi in factores secundi gradus reales numero p hoc modo

$$z^{2p}-bz^p+a^p=\overline{zz-2Cc.\frac{\varphi}{p}.z+a}.\overline{zz-2Cc.\frac{c+\varphi}{p}z+a}.$$

$$\overline{zz-2Cc.\frac{2c+\varphi}{p}.z+a}\ldots\ldots\overline{zz-2Cc.\frac{\overline{p-1}.c+\varphi}{p}z+a}.$$

Quod erat Inv.

Sed hypotheses aliquot accuratius evolvamus. Fiat primo $b=o$, ut habeatur binomium $z^{2p}+a^p=o$. In hac hypothesi, quando cosinus arcus $\varphi=o$, & sinus sit oportet positivus, arcus φ erit quadrans circuli. Quare vocata ut antea circumferentia $=c$, adeoque ejus quadrante $=\frac{c}{4}$, trinomia realia, in quæ binomium resolvitur, erunt hujusmodi

$$zz-2Cc.\frac{c}{4p}.z+a,\ zz-2Cc.\frac{5c}{4p}.z+a,\ zz-2Cc.\frac{9c}{4p}.z+a,$$

$$zz - 2Cc.\frac{13c}{4p}.z + a \ldots\ldots\ldots zz - 2Cc.\frac{\overline{4p-3}.c}{4p}z + a:$$

Sed de hoc casu paullo infra redibit sermo.

Ponamus modo $Cc.\varphi$, hoc est $\frac{b}{2.a^{\frac{p-1}{2}}} = a^{\frac{1}{2}}$, scilicet sinui toti. Proveniet $b = 2a^{\frac{p}{2}}$, & trinomium hanc formam induit $z^{2p} - 2a^{\frac{p}{2}}z^{p} + a^{p} = o$. Manifestum est $\varphi = o$: ergo ultimum trinomium resolvitur in sequentia trinomia secundi gradus

$$zz - 2Cc.\frac{o}{p}z + a,\ zz - 2Cc.\frac{c}{p}.z + a,\ zz - 2Cc.\frac{2c}{p}.z + a,$$

$$\ldots\ldots\ldots\ldots zz - 2Cc.\frac{\overline{p-1}.c}{p}.z + a.$$

Describe circulum, cujus radius $= a^{\frac{1}{2}}$, & facto (*Fig.* 1, 2) initio in puncto 1 divide totam circumferentiam in partes $2p$, ut semicircumferentia in partes p divisa reperiatur. In singulis divisionis punctis ordinatim appone numeros, ut figura manifestat. Liquet punctis omnibus, quæ signata sunt numeris imparibus, respondere cosinus quæsitos: arcus enim $13 = \frac{c}{p}$, $15 = \frac{2c}{p}$: atque ita deinceps.

Si recte animum advertas, cognosces, bis semper reperiri eumdem cosinum: nam cuilibet arcui minori quam semicircumferentia respondet arcus eadem major, qui præditus est eodem cosinu. Exipiendus tamen est arcus $= o$, cujus cosinus $= a^{\frac{1}{2}}$, & ubi p sit numerus par, excipiendus est arcus $= \frac{c}{2}$, hoc est semicircumferentia; horum enim arcuum cosinus reperiuntur semel. Verum arcus hujusmodi præbent trinomia $zz - 2a^{\frac{1}{2}}z + a$, $zz + 2a^{\frac{1}{2}}z + a$, quæ quadrata sunt, & quorum radices extrahi possunt. Quapropter satis erit dividere semicircumferentiam

tiam in partes p facto initio a puncto 1, & accipere cosinus omnium arcum desinentium in puncta signata numeris imparibus, & ex his efformare trinomia, quæ erunt hujusmodi

$zz - 2Cc.\frac{c}{p}.z + a$, $zz - 2Cc.\frac{2c}{p}.z + a$ & cetera. Nostrum itaque trinomium resolubile est in hæc trinomia elata ad potestatem quadraticam, addito semper trinomio $zz - 2a^{\frac{1}{2}}z + a$, & si p sit par, etiam trinomio $zz + 2a^{\frac{1}{2}}z + a$. Habemus ergo æquationem

$$z^{2p} - 2a^{\frac{p}{2}}z^{p} + a^{p} = \overline{zz - 2Cc.\frac{c}{p}.z + a}^{2} . \overline{zz - 2Cc.\frac{2c}{p}.z + a}^{2} . \overline{zz - 2Cc.\frac{3c}{p}.z + a \; \&ce.}^{2} . zz - 2a^{\frac{1}{2}}z + a . {}^{*}zz + 2a^{\frac{1}{2}}z + a.$$

Trinomium ultimum, cui opposui *, addendum non est, nisi p fuerit par. Igitur extracta radice habebimus

$$z^{p} - a^{\frac{p}{2}} = zz - 2Cc.\frac{c}{p}z + a . \overline{zz - 2Cc.\frac{2c}{p}z + a} . \overline{zz - 2Cc.\frac{3c}{p}.z + a \; \&ce.} . \overline{z - a^{\frac{1}{2}}} . {}^{*}\overline{z + a^{\frac{1}{2}}}.$$

Demum ponamus $Cc.\varphi$ hoc est $\frac{b}{2a^{\frac{p-1}{2}}} = -a^{\frac{1}{2}}$, scilicet sinui toti negative sumpto, & trinomium hoc nascetur

$z^{2p} + 2a^{\frac{p}{2}}z^{p} + a^{p} = 0$. Evidens est, arcum φ æquare circumferentiæ dimidium nempe $= \frac{c}{2}$. Quare habes trinomia, in quæ fit resolutio, nempe

$$zz - 2Cc.\frac{c}{2p}.z + a, \; zz - 2Cc.\frac{3c}{2p}.z + a, \; zz - 2Cc.\frac{5c}{2p}.z + a$$

.....

. . . . $zz - 2Cc.\overline{\frac{2p-1.c}{2p}}.z + a.$

Initio facto a puncto 1 integram circumferentiam divide in paries $2p$, & numeris naturalibus ordinatim signa puncta divisionis, ut factum est antea. Cosinus accipiendi sunt eorum arcuum, quorum termini a numeris paribus definiuntur. Nam arcus $12 = \frac{c}{2p}$, $14 = \frac{3c}{2p}$, atque ita de reliquis.

Hic quoque evenit, ut bis cosinus singuli sint capiendi; existunt enim semper duo arcus unus minor, alter major semicircumferentia, qui eumdem cosinum habent. Excipe tamen dimidium circumferentiæ, cujus cosinus non ingreditur in trinomia, nisi p fuerit impar: quo in casu quum cosinus $= -a^{\frac{1}{2}}$, resultabit trinomium $zz + 2a^{\frac{1}{2}}z + a$, quod est quadratum prædictum radice $z + a^{\frac{1}{2}}$. Quare satis est dividere semicircumferentiam in partes p, facto initio ab 1, accipere cosinus arcuum desinentium in numeros pares, & ex his formata trinomia elevare ad quadratum, quibus addendum est trinomium $zz + 2a^{\frac{1}{2}}z + a$, si p sit impar.

Hoc modo obtinemus æquationem

$$z^{2p} + 2a^{\frac{p}{2}}z^{p} + a^{p} = \overline{zz - 2Cc.\frac{c}{2p}.z + a}^{2}.\overline{zz - 2Cc.\frac{3c}{2p}.z + a}^{2}.$$

$$\overline{zz - 2Cc.\frac{5c}{2p}.z + a}^{2} \text{ \& ce. } * \overline{zz + 2a^{\frac{1}{2}}z + a}.$$

Signum * denotat trinomium non esse scribendum, nisi existente p impari. Extrahatur, radix quadrata

$$z^{p} + a^{\frac{p}{2}} = \overline{zz - 2Cc.\frac{c}{2p}.z + a}.\overline{zz - 2Cc.\frac{3c}{2p}.z + a}.$$

$$\overline{zz - 2Cc.\frac{5c}{2p}.z + a} \text{ \& ce. } * \overline{z + a^{\frac{1}{2}}}.$$

Ex

Ex his facillima, atque expeditissima fluit demonstratio celeberrimi theorematis cottesiani; quod tu ipse cognosces, ubi probatum fuerit sequens lemma. In circulo, cujus centrum C, radius CA $= a^{\frac{1}{2}}$, assumpto quolibet (*Fig.* 3, 4) puncto B, vocetur CB $= z$; agatur quælibet BD, & demittatur DE, quæ sit sinus arcus AD, fiat autem cosinus CE $= x$: ajo

BD $= \sqrt{zz - 2xz + a}$. Liquet ED$^2 = a - xx$, BE $= \pm z \mp x$, signa superiora valent in tertia, inferiora in quarta figura: ergo BE$^2 = zz - 2xz + xx$: ergo BD$^2 = zz - 2xz + xx + a = zz - 2xz + a$, & $- xx$

BD $= \sqrt{zz - 2xz + a}$. Quod erat Dem.

Deinceps arcus indicabo per numeros, a quibus (*Fig.* 1, 2) in figuris terminantur. Paullo ante probatum est

$$z^{2p} - 2a^{\frac{p}{2}} z^p + a^p = \overline{zz - 2Cc.0.z + a} . \overline{zz - 2Cc.13.z + a}$$
$\overline{zz - 2Cc.15.z + a}$ & ce., donec circumferentia integra exhauriatur. Seca CB $= z$, & fiet

$$z^{2p} - 2a^{\frac{p}{2}} z^p + a^p = B1^2 . B3^2 . B5^2 \ \&\ ce.,$$ & extracta radice $\pm z^p \pm a^{\frac{p}{2}} = B1 . B3 . B5$ & ce.

Probatum item est antea

$$z^{2p} + 2a^{\frac{p}{2}} z + a^p = \overline{zz - 2Cc.12.z + a} . \overline{zz - 2Cc.14.z + a}$$
$\overline{zz - 2Cc.16.z + a}$ & ce. donec exhausta fuerit omnis circumferentia: igitur

$$z^{2p} + 2a^{\frac{p}{2}} z^p + a^p = B2^2 . B4^2 . B6^2 \ \&\ ce.$$ extractaque radice

$$z^p + a^{\frac{p}{2}} = B2 . B4 . B6 \ \&\ ce.$$ Quod Erat D.

Similis constructio accommodari etiam potest trinomio $z^{2p} - bz^{p} + a^{p}$, quoties $\frac{bb}{4} < a^{p}$. Etenim seca arcum A D, (*Fig.* 5.) cujus cosinus $= \frac{b}{2 \cdot a^{\frac{p-1}{2}}}$, qui arcus erit minor quadrante, si b sit positiva, major, si b sit negativa. Hunc arcum divide in partes p, quarum prima sit A 1. Ex puncto 1 incipe dividere circumferentiam in partes p in punctis 2, 3, 4 & ce. Cosinus arcuum A 1, A 2, A 3 & ce. positi in trinomio $zz - 2xz + a$, exhibent trinomia realia, in quæ fit resolutio: igitur facta C B $= z$, actisque B 1, B 2, B 3 & ce., nanciscemur $z^{2p} - bz^{p} + a^{p} = \text{B}\,1^{2} \cdot \text{B}\,2^{2} \cdot \text{B}\,3^{2} \cdot$ & ce. Quod erat Inven.

Quamquam ea, quam tradidi, in trinomia realia *resolutio* nova non est, sed a pluribus demonstrata; tamen non injucunda, neque inelegans visa est methodus, qua eamdem deduco ab æquatione recipiente radicem cardanicam. Hujusce æquationis fortasse amplior est utilitas, atque usus: sed in præsentia rem persequi non vacat. Si hæc, quæ ad te scribo, fuerint tuo judicio probata, nonnihil otii, si quid dabitur, in hisce inquirendis consumam. Vale

Bononiæ postridie Kal. Septembris 1757. Corcyram ad Franciscum Benaleam.

FINIS OPUSCULORUM.

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Fig. 1.

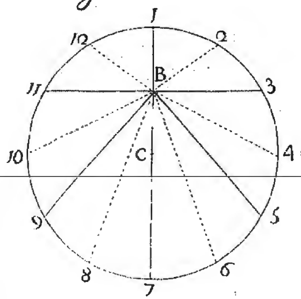


Fig. 2.

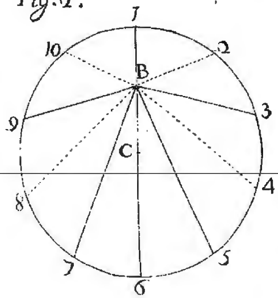


Fig. 3.

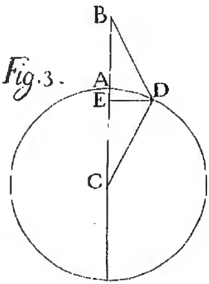


Fig. 4.

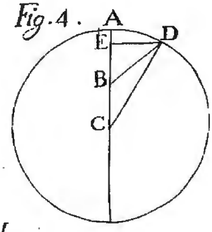
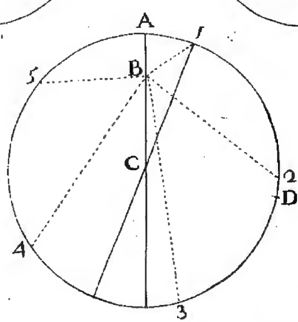


Fig. 5.



INDEX AUCTORUM

Quorum mentio fit in hoc volumine.

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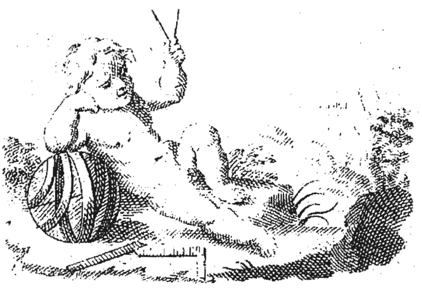

ROMUALDUS ROTA E SOCIETATE JESU

In Provincia Veneta Præpositus Provincialis.

CUm Librum, cui titulus = VINCENTII RICCATI Opusculorum ad res Physicas, & Mathematicas pertinentium Tomus Secundus =, a Patre Vincentio Riccati Nostræ Societatis Sacerdote conscriptum, aliquot ejusdem Societatis Theologi recognoverint, & in lucem edi posse probaverint, potestate nobis a R. P. N. Laurentio Ricci Præposito Generali ad id tradita, facultatem concedimus, ut Typis mandetur, si ita iis, ad quos pertinet, videbitur. Cujus rei gratia has litteras manu nostra subscriptas, & sigillo nostro munitas dedimus. Bononiæ die 9 Februarii 1761.

Romualdus Rota.

Vidit D. Joseph Maria Vidari Clericus Regularis Sancti Paulli, & in Ecclesia Metropolitana Bononiæ Pœnitentiarius pro Eminentissimo, & Reverendissimo Domino D. Vincentio Cardinali Malvetio Archiepiscopo Bononiæ, & S. R. I. Principe.

Die 25. Januarii 1761.

A. R. P. Carolus Maria Offredi Ordinis Theatinorum Publicus Universitatis Bononiæ Professor, & Sancti Officii Revisor Ordinarius videat pro Sancto Officio, & referat.

Fr. Tomas Maria de Angelis S.Off.Bonon. Generalis Inquisit.

Nihil orthodoxæ fidei contrarium, nihil bonis moribus repugnans continet liber inscriptus = *Opusculorum ad res Physicas, & Mathematicas pertinentium Tomus Secundus*, = cujus est auctor Vir Clarissimus, & in Mathematicis disciplinis versatissimus Pater Vincentius Riccati Soc. Jesu, quemque de mandato Reverendiss. P. Inquisitoris Generalis Bononiæ attente perlegi. Quapropter dignum censeo, ut publica luce donetur.

Bononiæ ex Domo S. Bartholomæi Apostoli prid. id. Feb. 1761.

D. Carolus Maria Offredi C. R. Lector Pub. & S. O. Revisor Ord.

Die 12. Februarii 1761.

Attenta superposita attestatione.

IMPRIMATUR.

Inquisitor Generalis Sancti Officii Bononiæ.